SPECIAL CLASSES OF EXCESSIVE FUNCTIONS SATISFYING HARNACK PRINCIPLE

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Let $\mathcal{H}^{\text{inf}}$ be the family of functions each of which is an infimum of arbitrary collection of positive harmonic functions on a bounded domain D in $\mathbf{R}^n$. $\mathcal{H}^{\text{inf}}$ is a convex cone of positive functions closed in the topology of pointwise convergence and closed under arbitrary infima. Every such function is excessive for Brownian motion in D and satisfies Harnack inequality.

The most important functions in the class are those defined by

$$s_a = \inf\{h : \ h > 0 \text{ harmonic}, h(a) = 1\}, \quad a \in D,$$

which give the best estimate for the constant in Harnack inequality. Those functions are explicitly computed for the unit ball and the halfspace. It is shown for Lipschitz domains, that the knowledge of functions s_a is enough to recover all positive harmonic functions. More precisely,

$$\lim_{a \to z} \frac{s_a(x)}{s_a(x_0)} = K(x, z)$$

for every $z \in \partial D$, where K is the kernel function for D at x_0.

Several examples of functions in $\mathcal{H}^{\text{inf}}$ are given; for instance, the expected exit time from D for the Brownian motion is a member of the family. A necessary condition sátisfied by the Riesz measure of any function in $\mathcal{H}^{\text{inf}}$ is obtained.

Results are proved in a more general setting which makes it possible to simultaneously treat both harmonic functions and positive solutions of the equation $Lu = 0$, where L includes uniformly elliptic operators in divergence form or Schrödinger operators. Positive harmonic functions for a certain class of diffusions are also included (with more restrictive results).

It is shown that positive harmonic functions determine Brownian motion up to time change. Moreover, any diffusion in a Lipschitz domain D whose killing distributions are equal to the exit distributions of Brownian motion from D is necessarily time changed Brownian motion.

CHAPTER 1
INTRODUCTION

Positive harmonic functions on a bounded domain D in $\mathbf{R}^n$ enjoy two important and well-known properties: They satisfy Harnack inequality and allow integral representation via a kernel function.

Harnack inequality is usually expressed in two forms. The first one is more local in nature and states that if a ball $B(x_0, r)$ is contained in D, then for any positive harmonic function h on D, and any $x \in B(x_0, r)$,

$$r^{n-2}\frac{r^2 - |x - x_0|^2}{(r + |x - x_0|)^n}\, h(x_0) \leq h(x) \leq r^{n-2}\frac{r^2 - |x - x_0|^2}{(r - |x - x_0|)^n}\, h(x_0). \tag{1.1}$$

The second form follows from the inequality above: For any compact subset K of D there is a constant $c = c_K$ such that for any positive harmonic function h on D

$$\frac{h(x)}{h(y)} \leq c \quad \text{for all} \quad x, y \in K \tag{1.2}$$

(see, e.g., [14, p. 31]).

Harnack inequality has numerous applications. We concentrate on the following two: (i) the infimum of any family of positive harmonic functions satisfies both (1.1) and (1.2), and (ii) any such infimum is continuous. The second property follows from the fact that every function satisfying (1.1) is automatically continuous at x.

Hence, one can form the family

$$\mathcal{H}^{\inf} = \{u : u = \inf_{\alpha} h_\alpha,\ h_\alpha \ \text{ positive and harmonic in } D\}$$

consisting of continuous functions satisfying Harnack inequality. Every function in $\mathcal{H}^{\inf}$ is trivially superaveraging, so $\mathcal{H}^{\inf}$ consists of continuous positive superharmonic

functions on D which satisfy Harnack inequality. Another way of expressing this is that functions in $\mathcal{H}^{\mathrm{inf}}$ are excessive for the Brownian motion in D and satisfy Harnack inequality.

In Chapter 2 we prove the following result about $\mathcal{H}^{\mathrm{inf}}$:

THEOREM 2.1 *$\mathcal{H}^{\mathrm{inf}}$ is a convex cone stable for arbitrary infima and closed in the topology of pointwise convergence.*

Now we recall the second essential property of positive harmonic functions, namely the integral representation.

Following [11] let D be a domain in $\mathbf{R}^n$ with the Green function G_D and ν a measure with compact support $\mathrm{Supp}\nu \subset D$. The function

$$K_\nu(x,y) = \frac{G_D(x,y)}{G_D\nu(y)}$$

is called the *Martin function* (or the *Martin kernel*) based on ν. There is a unique metrizable compactification D_M of D such that each Martin function K_ν has a continuous extension (denoted also by K_ν) to $D \times (D_M \setminus \mathrm{Supp}\nu)$, and $K_\nu(\cdot, y_1) = K_\nu(\cdot, y_2)$ if and only if $y_1 = y_2$. The boundary $\partial_M D = D_M \setminus D$ is called the *Martin boundary.* Let $\partial_M^\circ D$ denote the set of minimal points $z \in \partial_M D$. Then for each positive harmonic function h on D there is a unique measure μ on $\partial_M^\circ D$ such that

$$h = \int_{\partial_M^\circ D} K_\nu(\cdot, z)\mu(dz). \tag{1.3}$$

If the measure ν is the point mass at $x_0 \in D$, then we say that the Martin kernel is based at x_0. In this case the continuous extension of the Martin kernel to $\partial_M D$ means that as $y \to z$ in the Martin topology, where $y \in D$, $z \in \partial_M D$, we have

$$\lim_{y \to z} \frac{G_D(x,y)}{G_D(x_0,y)} = K_{x_0}(x,z). \tag{1.4}$$

For the unit ball $D = B(0,1)$ the Martin boundary is simply the Euclidean boundary of D, and the Martin kernel is the Poisson kernel

$$P(x,z) = \frac{1-|x|^2}{|z-x|^n}.$$

Inequality (1.1) for the unit ball

$$\frac{1-|x|^2}{(1+|x|)^n}\, h(0) \le h(x) \le \frac{1-|x|^2}{(1-|x|)^n}\, h(0)$$

reveals the connection with the Poisson kernel. In fact, it is derived from the representation formula. Moreover, this inequality is sharp in the sense that the bounds are attained for $h = P(\cdot, z)$ where $z = x/|x|$ or $-x/|x|$. In a sense, Harnack inequality is tailor-made for balls. It is not optimal for other domains.

To get the optimal lower bounds for an arbitrary domain D, it is natural to introduce the function

$$k(x,y) = \inf\{h(x) : h \text{ positive harmonic in } \ D,\ h(y) = 1\}, \quad y \in D. \qquad (1.5)$$

From Theorem 1 this function is continuous in x (in fact, it is jointly continuous; see Proposition 2.1) and gives the greatest lower bound for a positive harmonic function which is 1 at y. As in the case of the unit ball, this lower bound is attained by a Martin kernel: There exists $z = z(x,y)$ in the minimal Martin boundary $\partial_M^\circ D$ such that

$$k(x,y) = \frac{K(x,z)}{K(y,z)}$$

where K is based at some point x_0 in D (see Proposition 2.4). Therefore, the Martin kernel $K(x,z)$ completely determines function k.

For Lipschitz domains the converse is also true: Function k is sufficient to recover the Martin kernel. More precisely, let D be a bounded Lipschitz domain. Then the Martin boundary $\partial_M D$ of D is exactly the Euclidean boundary (see [15]),

and all boundary points are minimal. Let K be the Martin kernel based at $x_0 \in D$. In Theorem 2.2 we prove that if $x \in D$, and $z \in \partial D$, then

$$\lim_{y \to z} \frac{k(x,y)}{k(x_0,y)} = K(x,z). \tag{1.6}$$

This result can be regarded as an analogue of (1.3). The reason we restrict ourselves to Lipschitz domains is that our proof relies on the fact that if $z_1, z_2 \in \partial D$, $z_1 \neq z_2$, $x \in D$, $z \in \partial D$, then

$$\lim_{(x,z) \to (z_1,z_2)} K(x,z) = 0. \tag{1.7}$$

This is not true in general: There are examples of domains such that z_1, z_2 are distinct minimal boundary points on the Martin boundary, but $\lim K(x,z) = +\infty$ as $(x,z) \to (z_1, z_2)$ (see [19]). We note that (1.7) is also true for so-called "NTA" domains introduced in [16]. That the limit in (1.7) is zero follows from Lemma 5.2 in [16, p.103].

In Sections 4.1 and 4.2 we obtain explicit formulae for k in case D is the unit ball or the upper halfspace. We note that the similar formulae were also obtained in [17]. Computations rely on the Möbius and Kelvin transform. Those formulae show that k is *not* symmetric if dimension $n \geq 3$. For $n = 2$, k is symmetric, which is also true for some other domains in $\mathbf{R}^2$.

As we have mentioned above, $\mathcal{H}^{\text{inf}}$ is a convex cone of positive functions. It is natural to ask whether $\mathcal{H}^{\text{inf}}$ is a potential cone. Although general wisdom tells us the answer is no, it does not seem that potential-theoretic arguments can justify that, simply because in case $n = 1$ the answer is yes. In Section 4.4 we give a simple example for $n = 2$ that confirms that no potential theory is incorporated in $\mathcal{H}^{\text{inf}}$.

An intriguing question to us is, given a continuous excessive function on D, how can we recognize it as an infimum of positive harmonic functions? In Section 3.3 we give a partial answer: If $u = G_D\mu$ is a potential in $\mathcal{H}^{\text{inf}}$, then μ cannot

have a compact support (provided that D "has no holes"). This is, of course, very far from being sufficient. We do not know if any condition on a measure μ can be given to guarantee that $G_D\mu$ is in $\mathcal{H}^{\inf}$. The next example should provide some insight into the difficulties. Let D be the unit disc in $\mathbf{R}^2$, $z = (1,0) \in \partial D$, and $u = P(\cdot, z) \wedge 1$. We show in Section 4.5 that u is extremal in $\mathcal{H}^{\inf}$. It is easy to see that the Riesz measure of u is concentrated on the boundary S of the disc $B(x_0, 1/2)$ where $x_0 = (1/2, 0)$. Let A,B form any partition of S such that neither A nor B is compact. Then $u = G_D\mu_{|A} + G_D\mu_{|B}$, and, since u is extremal in $\mathcal{H}^{\inf}$, at least one of the potentials $G_D\mu_{|A}$, $G_D\mu_{|B}$ is not in $\mathcal{H}^{\inf}$.

In view of the above, it is not surprising that not many instances of functions in $\mathcal{H}^{\inf}$ are available. We give several examples in Section 4.3, the most interesting being the expected exit time of the Brownian motion from D.

The organization of the thesis is directed by an attempt to include more general situations. Topological properties of the cone $\mathcal{H}^{\inf}$ depend exclusively on the fact that positive harmonic functions on a domain form a closed convex cone with compact basis. Therefore, in Chapter 2, we study closed convex cone $\mathcal{H}$ with compact basis of positive continuous functions on a locally compact space. We introduce the function k as in (1.5) and the cone $\mathcal{H}^{\inf}$ of all infima of function from $\mathcal{H}$. We note that a similar approach can be found in [7].

In Section 2.2 we assume the existence of a kernel function K. We require K to have properties of the Martin kernel on a Lipschitz domain and postulate them in hypotheses (H_1)-(H_3). In this context we prove that (1.6) holds.

In Chapter 3 we give examples of cones with compact basis. Instead of positive harmonic functions for the Laplacian, one can consider positive solutions to $Au = 0$, where A is a uniformly elliptic operator in divergence form. By using results in [4] which parallel those obtained in [15], we prove that (1.6) is valid, where now K is

the kernel function for (A, D), D Lipschitz, and k defined by (1.5) for the cone of positive harmonic functions for A.

In Section 3.2 the same result is obtained for positive solutions of Schrödinger equation $-Au + qu = 0$, for q in Kato class with finite gauge and D Lipschitz. Here we use results from [5] and [9]. We note that equivalent statements can be obtained for positive solutions of the equation $\frac{1}{2}\Delta u + b \cdot \nabla u = 0$ by using results from [10]. The drift b must satisfy conditions given in that paper.

In Section 3.3 we use probabilistic methods and show that positive harmonic functions for certain diffusions also form a convex cone with compact basis. We rely on results from [21] and [22]. Of course, in this setting the kernel function satisfying our hypotheses (H_1)-(H_3) is not available.

In Chapter 4 we return to classical harmonic functions and obtain some formulae and examples described above. In addition, we obtain some (but not all) extremal elements in $\mathcal{H}^{\text{inf}}$ on the unit ball.

In Chapter 5 we investigate a diffusion on a bounded domain D which has the same class of harmonic functions as the Brownian motion in D. We show that such a diffusion is necessarily a time change of the Brownian motion. Moreover, if the diffusion dies as it approaches the boundary ∂D with the same distribution as the Brownian motion, it is again a time change.

CHAPTER 2
CONVEX CONES OF POSITIVE CONTINUOUS FUNCTION

2.1 Topological Properties

Let D be a locally compact topological space with countable basis. Then D is metrizable and let d denote a metric compatible with the topology on D. Let $\{K_n\}_{n=1}^{\infty}$ be an increasing sequence of compact sets covering D such that K_n lies in the interior of K_{n+1} for each $n \in \mathbf{N}$.

Let $\mathcal{C}(D)$ denote the vector space of all real-valued continuous functions on D, topologized by the separating family of seminorms

$$p_n(f) = \sup\{|f(x)| : x \in K_n\}.$$

The topology on $\mathcal{C}(D)$ is metrizable, and $\mathcal{C}(D)$ thus becomes a Fréchet space. Convergence in this topology is the uniform convergence on compact subsets of D. When we refer to any topological property of $\mathcal{C}(D)$, we always refer to this topology.

A subset B of $\mathcal{C}(D)$ is *bounded* if and only if for each compact subset K of D, there exists a constant C_K such that $|f(x)| \leq C_K$ for each $f \in B$ and each $x \in K$.

Let $A \subset D$ and $U \subset \mathcal{C}(D)$. U is said to be uniformly equicontinuous on A if for every $\epsilon > 0$ there exists $\delta > 0$ such that $|u(x) - u(y)| < \epsilon$ whenever $d(x, y) < \delta$, $x, y \in A$, $u \in U$. A family $U \subset \mathcal{C}(D)$ is *locally uniformly equicontinuous* if it is uniformly equicontinuous on each compact subset K of D.

We shall freely use Arzela-Ascoli theorem: A family $U \subset \mathcal{C}(D)$ is compact if and only if it is closed, bounded, and locally uniformly equicontinuous.

By $\mathcal{C}^+(D)$ we denote the family of strictly positive functions in D. Let $\mathcal{H}' \subset \mathcal{C}^+(D)$ and let $\mathcal{H} = \mathcal{H}' \cup \{0\}$, where 0 denotes the function which is identically zero.

We assume that $\mathcal{H}$ is a convex cone of functions, closed in $\mathcal{C}(D)$, having a *compact basis.* The last condition means that there exists a hyperplane $\mathcal{L}$ in $\mathcal{C}(D)$ such that the family $\mathcal{G} = \mathcal{L} \cap \mathcal{H}'$ is compact in $\mathcal{C}(D)$ and generates $\mathcal{H}$:

$$\mathcal{H} = \{\lambda u : \lambda \geq 0, u \in \mathcal{G}\}.$$

The hyperplane $\mathcal{L} = \{f \in \mathcal{C}(D) \ : < \phi, f >= a\}$, where ϕ is a continuous linear functional and $a \in \mathbf{R}$.

Having established the setting, we proceed by defining the family of all possible infima of functions in $\mathcal{H}$:

$$\mathcal{H}^{\inf} = \{u : u = \inf_{\alpha \in \mathcal{A}} u_\alpha, u_\alpha \in \mathcal{H}\}. \tag{2.1}$$

The main purpose of this section is to show that the family $\mathcal{H}^{\inf}$consists of strictly positive continuous functions (including zero) and that it is closed in the topology of pointwise convergence. The basic tool toward this goal is the function $k : D \times D \to \mathbf{R}$ defined by

$$k(x,y) = \inf_{u \in \mathcal{H}'} \frac{u(x)}{u(y)}. \tag{2.2}$$

This function gives the best estimate for $u(x)$ when $u(y)$ has some prescribed value. This will be made more precise later. We emphasize that, in general, k is *not* symmetric in x and y.

Note that

$$\inf_{u \in \mathcal{H}'} \frac{u(x)}{u(y)} = \inf_{u \in \mathcal{G}} \frac{u(x)}{u(y)}, \tag{2.3}$$

hence k could be computed using only functions in the basis $\mathcal{G}$.

Further, for x, y, z in D,

$$k(x,z)k(z,y) = \inf_{u\in\mathcal{H}'} \frac{u(x)}{u(z)} \inf_{u\in\mathcal{H}'} \frac{u(z)}{u(y)} \leq \inf_{u\in\mathcal{H}'} \frac{u(x)}{u(y)} = k(x,y). \tag{2.4}$$

By taking $x = y$ and using the trivial fact that $k(x,x) = 1$, we get

$$k(x,z)k(z,x) \leq 1, \quad x, z \in D. \tag{2.5}$$

<u>LEMMA 2.1</u> *For all x, y in D, $k(x,y) > 0$.*

PROOF: By (2.3), $k(x,y) = \inf_{u\in\mathcal{G}} \frac{u(x)}{u(y)}$. Hence, there is a sequence $\{u_n\} \subset \mathcal{G}$ such that $u_n(x)/u_n(y) \to k(x,y)$. By compactness of $\mathcal{G}$, there exist a subsequence $\{u_{n_j}\}$ and $u \in \mathcal{G}$ such that $u_{n_j} \to u$ in $\mathcal{C}(D)$. Therefore,

$$k(x,y) = \lim_{j\to\infty} \frac{u_{n_j}(x)}{u_{n_j}(y)} = \frac{u(x)}{u(y)} > 0,$$

since $u > 0$.□

Since all functions in $\mathcal{H}'$ are strictly positive, one can consider the family $\mathcal{G}^{-1} = \{1/u : u \in \mathcal{G}\}$ which is also contained in $\mathcal{C}^+(D)$. This family is compact in $\mathcal{C}(D)$. Indeed, given a sequence $\{1/u_n\}$ in $\mathcal{G}^{-1}$, compactness of $\mathcal{G}$ yields a subsequence $\{u_{n_j}\}$ and $u \in \mathcal{G}$ such that $u_{n_j} \to u$ in $\mathcal{C}(D)$. Hence, $1/u_{nj} \to 1/u$ pointwise, and uniform convergence follows from

$$|\frac{1}{u_{n_j}} - \frac{1}{u}| = \frac{|u - u_{n_j}|}{|u_{n_j}||u|}$$

and boundedness of the subsequence on compacts.

Let K be a compact subset of D. Since $\mathcal{G}$ is bounded, there is a constant $C = C_K$ such that $u(x) \leq C$ for each $u \in \mathcal{G}$ and each $x \in K$. Similarly, there is $c = c_K$ such that $1/u(x) \leq 1/c$ for each $1/u \in \mathcal{G}^{-1}$ and for each $x \in K$. Hence,

$$c \leq u(x) \leq C, \quad u \in \mathcal{G},\ x \in K. \tag{2.6}$$

<u>LEMMA 2.2</u> *For each $y \in K$, the function $x \mapsto k(x,y)$ is uniformly continuous on K.*

PROOF: Let $\epsilon > 0$. By the local uniform equicontinuity of $\mathcal{G}$, there exists $\delta > 0$ such that

$$|u(x) - u(x')| < c\epsilon, \quad x, x' \in K, \ d(x,x') < \delta, \ u \in \mathcal{G},$$

where c is the constant from (2.6). This can be written as $u(x) < u(x') + c\epsilon$ and $u(x') < u(x) + c\epsilon$. After dividing the first inequality by $u(y)$ and using $c/u(y) \leq 1$, we get

$$\frac{u(x)}{u(y)} \leq \frac{u(x')}{u(y)} + \epsilon.$$

By taking the infimum over all u in $\mathcal{G}$, it follows $k(x,y) \leq k(x',y) + \epsilon$. Similarly, $k(x',y) \leq k(x,y) + \epsilon$, i.e.,

$$|k(x,y) - k(x',y)| \leq \epsilon, \quad x, x' \in K, \ d(x,x') < \delta. \tag{2.7}$$

□

<u>LEMMA 2.3</u> *For each $x \in D$, the function $y \mapsto k(x,y)$ is continuous at x.*

PROOF: Without loss of generality, let us assume that $x \in \mathrm{Int}K$, where K is the compact set as above. Let $\epsilon > 0$ and choose $\delta > 0$ such that $|u(x) - u(y)| < c\epsilon$, $y \in K, \ d(x,y) < \delta, \ u \in \mathcal{G}$. After dividing by $u(y)$, it follows that $1-\epsilon \leq u(x)/u(y) \leq 1+\epsilon, \ y \in K, \ d(x,y) < \delta, \ u \in \mathcal{G}$. By taking the infimum over all $u \in \mathcal{G}$, we get $1 - \epsilon \leq k(x,y) \leq 1 + \epsilon$. Since $k(x,x) = 1$, the above reads

$$|k(x,y) - k(x,x)| \leq \epsilon, \quad y \in K, \ d(x,y) < \delta. \tag{2.8}$$

□

Now we are ready to establish the joint continuity of k.

Let $(x,y)\in D\times D$ and let $\{(x_n,y_n)\}_{n=1}^{\infty}$ be a sequence in $D\times D$ converging to (x,y). Then,

$$\frac{k(x_n,y_n)}{k(x,y)}=\frac{k(x_n,y_n)}{k(x_n,y)}\frac{k(x_n,y)}{k(x,y)}.$$

From (2.4),

$$k(y,y_n)\le\frac{k(x_n,y_n)}{k(x_n,y)}\le\frac{1}{k(y_n,y)}.$$

Therefore,

$$k(y,y_n)\frac{k(x_n,y)}{k(x,y)}\le\frac{k(x_n,y_n)}{k(x,y)}\le\frac{1}{k(y_n,y)}\frac{k(x_n,y)}{k(x,y)}. \tag{2.9}$$

As $n\to\infty$, $k(y,y_n)\to k(y,y)$ by Lemma 2.3, $k(x_n,y)\to k(x,y)$ by Lemma 2.2. Let $n\to\infty$ in (2.9); it follows

$$1\le\liminf_n\frac{k(x_n,y_n)}{k(x,y)}\le\limsup_n\frac{k(x_n,y_n)}{k(x,y)}\le 1,$$

which shows that $\lim_n k(x_n,y_n)=k(x,y)$. Thus, we have proved the following.

PROPOSITION 2.1 *The function $k: D\times D\to\mathbf{R}$ defined by (2.2) is strictly positive and (jointly) continuous.*

A family of nonnegative functions on D is usually said to satisfy *Harnack principle* if, for every compact subset K of D, there is a constant $C=C_K$ such that

$$u(x)\le Cu(y) \tag{2.10}$$

for all x and y in K, and all u in the family. The proposition above trivially implies the following

COROLLARY 2.1 *Let $\mathcal{H}$ be a closed convex cone as above. Then $\mathcal{H}$ satisfies Harnack principle.*

Proof: If K is compact, then the function k attains its strict positive minimum, say m, on $K \times K$. Then $u(x)/u(y) \geq k(x,y) \geq m$ for all $x, y \in K$ and any $u \in \mathcal{H}$.$\square$

Now we slightly change our point of view. Instead of getting the function k from a given cone $\mathcal{H}$, let us assume that we are given a (jointly) continuous function $k : D \times D \to (0, \infty)$ which is identically 1 on the diagonal, i.e., $k(x,x) = 1$ for each $x \in D$. Using k, we define the family $\mathcal{S}$ of all nonnegative functions $u : D \to \mathbf{R}$ satisfying

$$u(x) \geq k(x,y)u(y) \tag{2.11}$$

for all $x, y \in D$.

The following lemma contains some properties of the family $\mathcal{S}$.

Lemma 2.4 (i) *$\mathcal{S}$ is a convex cone of continuous, strictly positive functions (unless identically zero).*

(ii) *$\mathcal{S}$ is closed in the topology of pointwise convergence (and, hence, in $C(D)$).*

(iii) *$\mathcal{S}$ is stable under arbitrary infima and suprema.*

(iv) *If $\mathcal{T} \subset \mathcal{S}$ is bounded at some point $x \in D$, then $\mathcal{T}$ is bounded in $C(D)$.*

(v) *If $\mathcal{T} \subset \mathcal{S}$ is bounded in $C(D)$, then it is locally uniformly equicontinuous.*

(vi) *Let $x \in D$ and $\mathcal{S}_x = \{u \in \mathcal{S} : u(x) = 1\}$. Then $\mathcal{S}_x$ is a compact basis for $\mathcal{S}$.*

Proof: (i) If $x_n \to x$, then $\liminf_n u(x_n) \geq \liminf_n k(x_n, x)u(x) = u(x)$ $\geq \limsup_n k(x, x_n)u(x_n) = \limsup_n u(x_n)$, which gives continuity. The rest of (i), as well as (ii) and (iii) is trivial.

(iv) Let C be a constant such that $u(x) \leq C$ for each $u \in \mathcal{T}$. Let K be a compact set in D and $\alpha = \inf_{y \in K} k(x,y) > 0$. Then, for every $y \in K$,

$$C \geq u(x) \geq k(x,y)u(y) \geq \alpha u(y),$$

and so $u(y) \leq C/\alpha$, which proves that $\mathcal{T}$ is bounded in $\mathcal{C}(D)$.

(v) Let C be a constant such that $u(x) \leq C$ for each $u \in \mathcal{T}$ and each $x \in K$. For $\epsilon > 0$, there is $\delta > 0$ such that, for all (x_1, y_1) and (x_2, y_2) in $K \times K$ with $d_1((x_1,y_1),(x_2,y_2)) = d(x_1,x_2) + d(y_1,y_2) < \delta$, we have $|k(x_1,y_1) - k(x_2,y_2)| < \epsilon/C$. Hence, for $x, y \in K$ with $d(x,y) < \delta$, $|k(x,y) - 1| = |k(x,y) - k(x,x)| < \epsilon/C$, and $|k(y,x) - 1| < \epsilon/C$.

For $u \in \mathcal{T}$,

$$u(y) - u(x) \leq u(y) - k(x,y)u(y) = (1 - k(x,y))u(y),$$

and

$$u(x) - u(y) \leq u(x) - k(y,x)u(y) = (1 - k(y,x))u(x).$$

If $u(y) \geq u(x)$, then $|u(y) - u(x)| \leq |1 - k(x,y)|\ u(y) \leq C|1 - k(x,y)|$. Similarly if $u(x) \geq u(y)$, then $|u(y) - u(x)| \leq C|1 - k(y,x)|$.

Hence, $|u(y) - u(x)| \leq C \max\{|1 - k(x,y)|, |1 - k(y,x)|\} \leq \epsilon$, whenever $x, y \in K$, $d(x,y) < \delta$, and $u \in \mathcal{T}$.

(vi) By definition, $\mathcal{S}_x$ is bounded at x. Hence (iv) implies boundedness in $\mathcal{C}(D)$, which, by (v), yields local uniform equicontinuity. Trivially, $\mathcal{S}_x$ is closed, so Arzela-Ascoli theorem gives compactness. That $\mathcal{S}_x$ is a basis for $\mathcal{S}$ is obvious. □

To recapitulate: Starting from a strictly positive, continuous function k satisfying $k(x,x) = 1$, we have obtained a closed convex cone with compact basis of strictly positive (unless zero) continuous functions. Some care must be exercised, because for some functions k we get trivial cones. For example, if k is symmetric and $k(x,y) > 1$

for some $x, y \in D$, the only function which satisfies (2.11) is zero. If k is identically 1, we obtain only nonnegative constants. On the other hand, if k also satisfies condition (2.4), then for every $y \in D$, the function $x \mapsto k(x,y)$ is in $\mathcal{S}$. A non-trivial example of such a function is $k(x,y) = \exp(-d(x,y))$.

Let $\mathcal{T}$ be a closed convex cone contained in $\mathcal{S}$. Then $\mathcal{T}_x = \{u \in \mathcal{T} : u(x) = 1\}$ is a compact basis for $\mathcal{T}$. Besides being closed in the topology of uniform convergence on compacts, $\mathcal{T}$ is also closed in the pointwise topology. Indeed, if $\{u_n\}_{n=1}^{\infty}$ is a sequence in $\mathcal{T}$ such that $u_n(x) \to u(x)$ for every $x \in D$, then $\{u_n(x_0)\}_{n=1}^{\infty}$ is a bounded sequence, x_0 being any point in D. By (iv) and (v) of Lemma 2.4, $\{u_n\}_{n=1}^{\infty}$ is bounded in $\mathcal{C}(D)$ and locally uniformly equicontinuous, hence relatively compact. Therefore, there exist a subsequence $\{u_{n_j}\}$ and a function $v \in \mathcal{C}(D)$ such that $u_{n_j} \to v$ in $\mathcal{C}(D)$. Since $\mathcal{T}$ is closed, $v \in \mathcal{T}$. But $u_{n_j} \to u$ pointwise, so $u = v \in \mathcal{T}$.

Let us now go back to the original setting: $\mathcal{H}$ is a closed convex cone of strictly positive (unless zero) continuous functions, having a compact basis $\mathcal{G}$. The function $k : D \times D \to (0, \infty)$ is defined as in (2.2). Using this k, we form the cone $\mathcal{S}$ of functions satisfying (2.11). Trivally, $u(x) \geq k(x,y)u(y)$ for each $u \in \mathcal{H}$, and so $\mathcal{H} \subset \mathcal{S}$.

Let $\mathcal{I}_1 = \{u_1 \wedge u_2 \wedge \ldots \wedge u_n : u_j \in \mathcal{H},\ j = 1, 2, \ldots, n,\ n \in \mathrm{N}\}$ and $\mathcal{I} = \mathrm{Cl}\, \mathcal{I}_1$ be the closure of $\mathcal{I}_1$ in $\mathcal{C}(D)$. Both $\mathcal{I}_1$ and $\mathcal{I}$ are contained in $\mathcal{S}$. $\mathcal{I}$ is a closed convex cone, stable under finite minima. Convexity follows from the equality

$$(u_1 \wedge \ldots \wedge u_n) + (v_1 \wedge \ldots \wedge v_m) = \min\{(u_i + v_j) : i = 1, \ldots, n,\ j = 1, \ldots, m\}.$$

Obviously, $\mathcal{I}$ is the smallest closed convex cone stable under finite minima containing $\mathcal{H}$. The family $\mathcal{I}_{x_0} = \{u \in \mathcal{I} : u(x_0) = 1\}$, $x_0 \in D$, is compact and $\mathcal{I}$ is closed for pointwise limits. Furthermore, $\mathcal{I}$ is stable for countable infima. Indeed,

if $\{u_n\}_{n=1}^{\infty}$ is a sequence in $\mathcal{I}$ and $u = \inf u_n$, let $v_n = u_1 \wedge \ldots \wedge u_n \in \mathcal{I}$. Since $u = \lim_n v_n$, u is in $\mathcal{I}$.

Now we recall that the family of all possible infima of functions in $\mathcal{H}$ was defined by (2.1) and was denoted by $\mathcal{H}^{\inf}$. Since each function in $\mathcal{H}$ satisfies (2.11), the same is true for functions in $\mathcal{H}^{\inf}$. Hence $\mathcal{H}^{\inf} \subset \mathcal{S}$ and, in particular, each function in $\mathcal{H}^{\inf}$ is continuous.

<u>LEMMA 2.5</u> *Each function in $\mathcal{H}^{\inf}$ is an infimum of a countable family of functions in $\mathcal{H}$.*

PROOF: Let $u = \inf_{\alpha \in \mathcal{A}} u_\alpha$, $u_\alpha \in \mathcal{H}$. By Choquet's lemma (e.g.,[14, p.34]), there exists a countable set $\mathcal{A}_0 \subset \mathcal{A}$ with the property that if g is a lower semicontinuous function on D such that $g \leq \inf_{\alpha \in \mathcal{A}_0} u_\alpha$, then $g \leq u$. Take $g = \inf_{\alpha \in \mathcal{A}_0} u_\alpha$ and note that g is continuous. Then $\inf_{\alpha \in \mathcal{A}_0} u_\alpha \leq u$, and since the reverse inequality trivially holds, $u = \inf_{\alpha \in \mathcal{A}_0} u_\alpha$.□

Let us temporarily introduce the following notation:

$$\mathcal{J} = \{u : D \to \mathbf{R} : u = \lim_n u_n,\ u_1 \leq u_2 \leq \ldots, u_n \in \mathcal{H}^{\inf}\}.$$

Each function in $\mathcal{J}$ satisfies (2.11), so $\mathcal{J} \subset \mathcal{S}$.

In the following three lemmas, we show that all three families, $\mathcal{H}^{\inf}$, $\mathcal{J}$, and $\mathcal{I}$ are equal.

<u>LEMMA 2.6</u> $\mathcal{J} = \mathcal{I}$

PROOF: By Lemma 2.5, each function in $\mathcal{H}^{\inf}$ is a countable infimum of functions in $\mathcal{H}$, hence $\mathcal{H}^{\inf} \subset \mathcal{I}$. Since $\mathcal{I}$ is closed for pointwise convergence, it follows that $\mathcal{J} \subset \mathcal{I}$.

Conversely, let $u \in \mathcal{I}$. Then $u = \lim_n u_n$, where $u_n \in \mathcal{I}_1 \subset \mathcal{H}^{\inf}$. Let $v_k = \inf_{n \geq k} u_n$, for $k \in \mathrm{N}$. Then $v_k \in \mathcal{I}$, $v_k \in \mathcal{H}^{\inf}$, and $\{v_k\}$ is an increasing sequence. Hence, $v = \lim_k v_k \in \mathcal{J}$. But

$$v = \sup_k v_k = \liminf_n u_n = u,$$

so $u \in \mathcal{J}$. $\square$

<u>LEMMA 2.7</u> *Let $u \in \mathcal{H}^{\inf}$ and $x \in D$. Then there exists $v \in \mathcal{H}$ such that $v \geq u$ in D and $v(x) = u(x)$.*

PROOF: Let $u = \inf u_\alpha$. There is a sequence $\{u_n\} \subset \{u_\alpha\}$ such that $u_n(x) \downarrow u(x)$. Thus $\{u_n\}$ is bounded at x, and, by (iv) and (v) of Lemma 2.4, it is relatively compact. Therefore, there is a subsequence $\{u_{n_j}\}$ and $v \in \mathcal{H}$ such that $u_{n_j} \to v$ in $\mathcal{C}(D)$. In particular, $u_{n_j}(x) \to v(x)$, so $u(x) = v(x)$. Since $u_n \geq u$ in D for each $n \in \mathrm{N}$, it follows that $v \geq u$ in D. $\square$

<u>LEMMA 2.8</u> $\mathcal{J} = \mathcal{H}^{\inf}$

PROOF: Let $u = \uparrow \lim_n u_n \in \mathcal{J}$, $u_n \in \mathcal{H}^{\inf}$. Fix $x \in D$. By Lemma 2.7, for each $n \in \mathrm{N}$ there is $v_n \in \mathcal{H}$ such that $v_n \geq u_n$ in D and $v_n(x) = u_n(x)$. Hence $\{v_n\}$ is bounded at x and therefore relatively compact. Thus, there exists a subsequence $\{v_{n_j}\}$ and $v^x \in \mathcal{H}$ such that $v_{n_j} \to v^x$ in $\mathcal{C}(D)$. Furthermore,

$$v^x(x) = \lim_j v_{n_j}(x) = \lim_j u_{n_j}(x) = u(x).$$

Since $v_{n_j} \geq u_{n_j}$ in $\mathcal{C}(D)$, it follows that $v^x \geq u$ in D.

Let $R_u = \inf\{v \in \mathcal{H} : v \geq u\}$. Then $R_u \in \mathcal{H}^{\inf}$ and $R_u \geq u$. But, for $x \in D$, $v^x \geq u$ and $v^x(x) = u(x)$. Hence, $R_u(x) \leq u(x)$. This proves $R_u = u$, i.e., $u \in \mathcal{H}^{\inf}$. $\square$

By putting together previous lemmas, we obtain the following.

<u>THEOREM 2.1</u> *Let $\mathcal{H}$ be a closed convex cone of strictly positive (unless zero) continuous functions. Assume that $\mathcal{H}$ has a compact basis. Then the family*

$$\mathcal{H}^{\inf} = \{u : u = \inf_{\alpha \in \mathcal{A}} u_\alpha,\ u_\alpha \in \mathcal{H}\}$$

is a closed convex cone, stable for arbitrary infima and closed in the topology of pointwise convergence.

<u>Remarks:</u> (i) Let $\mathcal{H}_y = \{u \in \mathcal{H} : u(y) = 1\}$, $y \in D$. Then $\mathcal{H}_y$ is a compact basis for $\mathcal{H}$. Therefore, by (2.2),

$$k(x,y) = \inf_{u \in \mathcal{H}_y} \frac{u(x)}{u(y)} = \inf\{u(x) : u \in \mathcal{H},\ u(y) = 1\}.$$

Let us denote the function above by $s_y(x)$. This function evaluated at x gives the greatest lower bound on numbers $u(x)$ for all $u \in \mathcal{H}$, given that $u(y) = 1$. Moreover, this bound is attained in the following sense: For given $x \in D$, there exists $v \in \mathcal{H}$ (depending on x and y) such that $v \geq s_y$ and $v(x) = s_y(x)$ (see Lemma 2.7).
(ii) $\mathcal{H}^{\inf}$ is a closed convex cone, and $(\mathcal{H}^{\inf})_y$ is a compact basis. One can consider the function

$$\overline{k}(x,y) = \inf_{u \in \mathcal{H}^{\inf}} \frac{u(x)}{u(y)}.$$

But this function is equal to $k(x,y)$, so nothing new is achieved. Note that $\mathcal{H}^{\inf}$ also satisfies Harnack principle.

In the sequel we consider some other properties of the cone $\mathcal{H}^{\inf}$. First we establish a large class of functions which operate on $\mathcal{H}^{\inf}$.

<u>PROPOSITION 2.2</u> *Assume that* $1 \in \mathcal{H}^{\inf}$. *If* $\phi : (0,\infty) \to \mathbf{R}$ *is a positive, increasing, concave function, then* $\phi \circ u \in \mathcal{H}^{\inf}$ *for each* $u \in \mathcal{H}^{\inf}$.

PROOF: ϕ is an infimum of linear functions $l(t) = at + b,\ \ a \geq 0,\ b \geq 0$. If $u \in \mathcal{H}^{\text{inf}}$, then $l \circ u = au + b$ is again in $\mathcal{H}^{\text{inf}}$. Let $\phi = \inf_{j \in J} l_j$. Then

$$\phi \circ u = (\inf_{j \in J} l_j) \circ u = \inf_{j \in J}(l_j \circ u) \in \mathcal{H}^{\text{inf}}.$$

□

Another operation which leaves $\mathcal{H}^{\text{inf}}$ invariant is integration. Let Σ be a compact topological space with the Borel σ-algebra $\mathcal{B}$. Assume that $u : D \times \Sigma \to \mathbf{R}_+$ has the following two properties:

(i) $x \mapsto u(x, \sigma) \in \mathcal{H}^{\text{inf}}$ for each $\sigma \in \Sigma$,

(ii) $\sigma \mapsto u(x, \sigma)$ is continuous for each $x \in D$.

Let μ be a positive Radon measure in $(\Sigma, \mathcal{B})$. By Theorem 12.11 in [6], the cone generated by point mass measures $\{\epsilon_\sigma\}$, $\sigma \in \Sigma$, is dense in the cone of all positive Radon measures $\mathcal{M}^+(\Sigma)$ (in vague topology). Hence, there is a sequence $\{\mu_n\} \subset \mathcal{M}^+(\Sigma)$ of measures of the type

$$\mu_n = \sum_{i=1}^{k} a_i \epsilon_{\sigma_i}, \quad a_i > 0,$$

which converges to μ. Thus, for every $f \in \mathcal{C}(\Sigma)$,

$$\int_\Sigma f\ d\mu_n \ \to\ \int_\Sigma f\ d\mu.$$

In particular, for each $x \in D$,

$$\int_\Sigma u(x, \sigma)\mu_n(d\sigma) \ \to\ \int_\Sigma u(x, \sigma)\mu(d\sigma).$$

But,

$$\int_\Sigma u(x, \sigma)\mu_n(d\sigma) = \sum_{i=1}^{k} a_i u(x, \sigma) \ \in\ \mathcal{H}^{\text{inf}},$$

and so $\int_\Sigma u(x,\sigma)\mu(d\sigma) \in \mathcal{H}^{\text{inf}}$, because $\mathcal{H}^{\text{inf}}$ is closed for pointwise convergence.

Let Σ be a locally compact space with countable basis, and let $\{\Sigma_n\}$ be an increasing sequence of compact subsets of Σ such that $\Sigma = \cup_n \Sigma_n$. If μ is a positive Radon measure on the Borel σ-algebra, then for $\mu_n = \mu_{|\Sigma_n}$ we have that

$$\int_{\Sigma_n} u(x,\sigma)\mu_n(d\sigma) = \int_\Sigma 1_{\Sigma_n} u(x,\sigma)\mu(d\sigma) \in \mathcal{H}^{\text{inf}}.$$

The Lebesgue monotone convergence theorem and pointwise closedness of $\mathcal{H}^{\text{inf}}$ imply that $\int_\Sigma u(x,\sigma)\mu(d\sigma) \in \mathcal{H}^{\text{inf}}$ (unless identically $+\infty$).

Let us assume now that the cone $\mathcal{H}$ contains positive constants and separates points. The family $\mathcal{D} = \mathcal{H}^{\text{inf}} - \mathcal{H}^{\text{inf}}$ is a vector lattice of continuous functions in D, containing all constants and separating points. By the Stone-Weierstrass theorem, $\mathcal{D}$ is dense in $\mathcal{C}(D)$. Moreover, if $f \in \mathcal{C}(D)$ is bounded by M, and $u_n \to f$ where $u_n \in \mathcal{D}$, then $v_n = (u_n \wedge M) \vee (-M) \in \mathcal{D}$ and $v_n \to f$ in $\mathcal{C}(D)$ boundedly.

Therefore, if μ is a positive Radon measure on the Borel σ-algebra on D and $f \in \mathcal{C}(D)$ with compact support, then the Lebesgue convergence theorem implies that $v_n \to f$ in $\mathcal{L}^p(D,\mu)$, $1 \leq p < \infty$, where $\{v_n\}$ is a bounded sequence of functions in $\mathcal{D}$ converging to f in $\mathcal{C}(D)$. Since continuous functions with compact support are dense in $\mathcal{L}^p(D,\mu)$, it follows that $\mathcal{D}$ is dense in $\mathcal{L}^p(D,\mu)$.

For the next consideration we assume that the cone $\mathcal{H}$ is linearly separating: For $x, y \in D$ and $\lambda \geq 0$, there exists $u \in \mathcal{H}$ such that $u(x) \neq \lambda u(y)$.

Let $\mathcal{C}_0(D)$ denote the set of continuous functions that vanish at infinity. We shall assume that the family $\mathcal{H}_0^{\text{inf}} = \mathcal{H}^{\text{inf}} \cap \mathcal{C}_0(D)$ is nonempty. Let $u \in \mathcal{H}_0^{\text{inf}}$ and, without loss of generality, we assume that $u(x_0) = 1$ where x_0 is some fixed point in D. For given ϵ, let K be a compact set such that $u(x) \leq \epsilon$ in $D \setminus K$. Then

$$\epsilon \geq u(x) \geq k(x,x_0)u(x_0) = s_{x_0}(x)$$

for $x \in D \setminus K$, and so $s_{x_0} \in \mathcal{H}_0^{\inf}$. From inequality (2.4) it follows that $s_y(x) \leq s_{x_0}(x)/s_{x_0}(y)$ and, therefore, $s_y \in \mathcal{H}_0^{\inf}$ for every $y \in D$.

Note that $\mathcal{H}_0^{\inf}$ is also a convex cone stable for arbitrary infima. If $u \in \mathcal{H}^{\inf}$ and v is an element of $\mathcal{H}_0^{\inf}$, then $u_n = (nv) \wedge u \in \mathcal{H}_0^{\inf}$, for each $n \in \mathbf{N}$, and $u_n \uparrow u$. Hence, every function in $\mathcal{H}^{\inf}$ is an increasing limit of functions in $\mathcal{H}_0^{\inf}$. In particular, $\mathcal{H}_0^{\inf}$ is also linearly separating.

Following [2] we say that a convex cone $\mathcal{P} \subset \mathcal{C}(D)$ is a *function cone* if $\mathcal{P}$ satisfies the following three conditions:

(F_1) There exists $v \in \mathcal{P}$ such that $v > 0$;

(F_2) $\mathcal{P}^+$ is linearly separating;

(F_3) $\mathcal{P}$ is *adapted*, i.e., for each $v \in \mathcal{P}$ there exist $u \in \mathcal{P}$ such that for each $\epsilon > 0$, there exists a compact set K in D with the property that $v(x) \leq \epsilon u(x)$ for every $x \in D \setminus K$.

LEMMA 2.9 *$\mathcal{H}_0^{\inf}$ is a function cone.*

PROOF: Both (F_1) and (F_2) follow from assumptions. For $v \in \mathcal{H}_0^{\inf}$, let $u = \sqrt{v}$ and, for $\epsilon > 0$, let $K = \{x \in D : \sqrt{v(x)} \geq \epsilon\}$. From Proposition 2.2, $u \in \mathcal{H}^{\inf}$, and so $u \in \mathcal{H}_0^{\inf}$. Since $\sqrt{v} \in \mathcal{C}_0(D)$, K is compact. Further, for $x \in D \setminus K$, $v(x) < \epsilon\sqrt{v(x)} = \epsilon u(x)$ which proves (F_3). □

It is not difficult to see that $\mathcal{H}^{\inf}$ need not be a function cone. One explicit example will be given in Chapter 4.

As it should be, reasonable functionals on $\mathcal{H}^{\inf}$ are given by measures. Let Φ be an additive, positively homogeneous, increasing functional on $\mathcal{H}^{\inf}$. Then $\Phi_{|\mathcal{H}_0^{\inf}}$ has the same properties. By one of the numerous Choquet's results, there exists a unique measure μ on the Borel σ-algebra of D, such that $\Phi(v) = \int_D v \, d\mu$ for every $v \in \mathcal{H}_0^{\inf}$ (e.g., [2, p.17, Prop.1.4]).

If $u \in \mathcal{H}^{\text{inf}}$, let $\{u_n\}$ be an increasing sequence of functions in $\mathcal{H}_0^{\text{inf}}$ such that $u = \uparrow \lim_n u_n$. Then

$$\Phi(u) = \lim_n \Phi(u_n) = \lim_n \int_D u_n \, d\mu = \int_D u \, d\mu.$$

This proves the following.

<u>Proposition 2.3</u> *Let $\Phi : \mathcal{H}^{\text{inf}} \to \mathbf{R}$ be an additive, positively homogeneous, increasing functional. Then there exists a unique measure μ on D such that $\Phi(u) = \int u \, d\mu$ for every $u \in \mathcal{H}^{\text{inf}}$.*

A function $u \in \mathcal{H}^{\text{inf}}$ is said to be *extremal* if $u = v + w$ for $v, w \in \mathcal{H}^{\text{inf}}$ implies that v and w are proportional to u, i.e., there is $\lambda > 0$ such that $v = \lambda u$, $w = (1-\lambda)u$. We show that for each $y \in D$ the function $s_y = k(\cdot, y)$ is extremal in $\mathcal{H}^{\text{inf}}$.

If $s_y = v + w$ where $v, w \in \mathcal{H}^{\text{inf}}$, then $1 = s_y(y) = v(y) + w(y)$. Further, for $x \in D$, $v(x) \geq k(x,y)v(y)$ and $w(x) \geq k(x,y)w(y)$. Therefore,

$$s_y(x) = v(x) + w(x) \geq s_y(x)v(y) + s_y(x)w(y) = s_y(x).$$

Hence, $v(x) = v(y)s_y(x)$ and $w(x) = w(y)s_y(x)$. Since this holds for each $x \in D$, we have $v = v(y)s_y$ and $w = w(y)s_y$.

We will show in Chapter 4 that, in general, there are more extremal functions than just s_y, $y \in D$.

2.2 Kernel Function

Let D be a topological space having the same properties as in the previous section. In addition, we assume that D is contained in a compact metrizable space denoted by $\overline{D}$, such that D is the interior of $\overline{D}$ and the metric of $\overline{D}$ restricted to D is d. Let ∂D denote $\overline{D} \setminus D$; we call ∂D the boundary of D.

Let $\mathcal{H}$ be a closed convex cone in $\mathcal{C}^+(D)$ containing the function identically equal to zero. We also assume that there is $u_0 \in \mathcal{H}$ satisfying $m \leq u_0 \leq M$ for some strictly positive constants m and M. Let x_0 be an arbitrary, but fixed point in D.

In this section we shall assume the existence of a function on $D \times \partial D$, which we call the *kernel function.* In the next chapter, it will be shown that such function exists for a broad class of convex cones.

The basic hypothesis is

(H_1) There exists a function $K : D \times \partial D \to \mathbf{R}$ such that

(i) for each $z \in \partial D$, $x \mapsto K(x,z)$ belongs to $\mathcal{H}_{x_0}$,

(ii) for each $x \in D$, $z \mapsto K(x,z)$ is continuous on ∂D.

Before we give the second hypothesis, let us note the following simple fact: If μ is a finite measure on the Borel subsets of ∂D, then the function u defined by

$$u(x) = \int_{\partial D} K(x,z)\mu(dz)$$

is in $\mathcal{H}$. This follows from one of the results of Section 2.1 (see p.19). We shall assume that all functions in $\mathcal{H}$ arise in this way.

(H_2) For each $u \in \mathcal{H}$, there exists a unique Borel measure μ on ∂D such that

$$u(x) = \int_{\partial D} K(x,z)\mu(dz).$$

Note: Since $K(x_0, z) = 1$ it follows that $u(x_0) = \mu(\partial D)$, so μ is a finite measure.

Let us recall the notation from the previous section: For $a \in D$, $s_a(x) = \inf\{u(x) : u \in \mathcal{H}, u(a) = 1\}$. Let

$$\mathcal{H}_{a,x} = \{u \in \mathcal{H} : u(a) = 1,\ u(x) = s_a(x)\}, \quad a, x \in D. \tag{2.12}$$

From the Remark following Theorem 2.1, it follows that $\mathcal{H}_{a,x}$ is nonempty. It is also convex, compact and closed for pointwise convergence. Since every function in $\mathcal{H}$ is representable by the kernel function, it is reasonable to expect that $\mathcal{H}_{a,x}$ contains functions of the form $x \mapsto K(x,z)$, $z \in \partial D$ (properly normalized). We show that this is true.

Let $u \in \mathcal{H}_{a,x}$ and μ the measure representing u. Then

$$s_a(x) = u(x) = \int_{\partial D} K(x,z)\mu(dz). \tag{2.13}$$

Assume that Δ' and Δ'' are disjoint Borel subsets of ∂D, such that $\Delta' \cup \Delta'' = \partial D$ and $\mu(\Delta') > 0$, $\mu(\Delta'') > 0$. Let $\alpha' = \int_{\Delta'} K(a,z)\mu(dz)$ and $\alpha'' = \int_{\Delta''} K(a,z)\mu(dz)$. Then both α' and α'' are strictly positive, so one can define measures ν' and ν'' on ∂D by

$$\nu' = \frac{1}{\alpha'}\mu_{|\Delta'} \quad \text{and} \quad \nu'' = \frac{1}{\alpha''}\mu_{|\Delta''}.$$

Let

$$u' = \int_{\partial D} K(\cdot,z)\nu'(dz) \quad \text{and} \quad u'' = \int_{\partial D} K(\cdot,z)\nu''(dz).$$

Then u' and u'' are in $\mathcal{H}$ and a simple computation shows that $u'(a) = u''(a) = 1$. Hence, $u'(x) \geq s_a(x)$ and $u''(x) \geq s_a(x)$. Therefore, since $\alpha' + \alpha'' = 1$,

$$u(x) = \alpha' u'(x) + \alpha'' u''(x) \geq \alpha' s_a(x) + \alpha'' s_a(x) = s_a(x) = u(x).$$

This implies that $u'(x) = u''(x) = s_a(x)$. Hence, u' and u'' are in $\mathcal{H}_{a,x}$ and representing measures have smaller support.

<u>PROPOSITION 2.4</u> *For $a \in D$ and $x \in D$, there exists $z = z(a,x) \in \partial D$ such that*

$$\frac{K(\cdot,z)}{K(a,z)} \in \mathcal{H}_{a,x}.$$

PROOF: Let $u \in \mathcal{H}_{a,x}$ and $u = \int_{\partial D} K(\cdot,\zeta)\mu(d\zeta)$. If μ is a multiple of a point mass at z, there is nothing to prove. If μ charges some point $z \in \partial D$, we take $\Delta' = \{z\}$ in the construction preceding the statement. Then the function u' from above is precisely $K(\cdot,z)/K(a,z)$.

So we assume that μ does not charge points. Let $z \in \mathrm{Supp}\mu$. Then μ charges every neighborhood of z in ∂D. Let $\{\Delta_n\}$ be a decreasing sequence of neighborhoods of z in ∂D which shrink to z, i.e., $\cap_{n=1}^{\infty}\Delta_n = \{z\}$. There is $n \in \mathbf{N}$ such that $\mu(\partial D \setminus \Delta_n) > 0$, since otherwise μ would be concentrated at z. We assume, without loss of generality, that this is true for every $n \in \mathbf{N}$.

Let

$$\alpha_n = \int_{\Delta_n} K(a,\zeta)\mu(d\zeta), \quad \nu_n = \frac{1}{\alpha_n}\mu_{|\Delta_n} \quad \text{and} \quad u_n = \int_{\partial D} K(\cdot,\zeta)\nu_n(d\zeta).$$

Then $u_n \in \mathcal{H}_{a,x}$ for each $n \in \mathbf{N}$.

We claim that the sequence of measures $\{\nu_n\}$ is bounded. First note that

$$\nu_n(\partial D) = \frac{\mu(\Delta_n)}{\alpha_n} = \frac{\mu(\Delta_n)}{\int_{\Delta_n} K(a,\zeta)\mu(d\zeta)}. \tag{2.14}$$

Since $\zeta \mapsto K(a,\zeta)$ is continuous on ∂D, there is $n_0 \in \mathbf{N}$ such that for every $\zeta \in \Delta_{n_0}$

$$\frac{1}{2}K(a,z) \leq K(a,\zeta) \leq 2K(a,z).$$

Integrate the above inequalities with respect to μ over Δ_n, $n \geq n_o$, to get

$$\frac{1}{2}K(a,z)\mu(\Delta_n) \leq \int_{\Delta_n} K(a,\zeta)\mu(d\zeta) \leq 2K(a,z)\mu(\Delta_n).$$

After dividing $\mu(\Delta_n)$ by each of the terms above, we obtain

$$\frac{1}{2K(a,z)} \leq \frac{\mu(\Delta_n)}{\int_{\Delta_n} K(a,\zeta)\mu(d\zeta)} \leq \frac{2}{K(a,z)}, \quad n \geq n_0.$$

But the middle term is, by (2.14), equal to $\nu_n(\partial D)$, which proves the claim.

Boundedness of the sequence $\{\nu_n\}$ implies the existence of a weakly convergent subsequence. We may assume, without loss of generality, that $\{\nu_n\}$ weakly converges to a positive Borel measure ν on ∂D. Let

$$v = \int_{\partial D} K(\cdot,\zeta)\nu(d\zeta) = \lim_n \int_{\partial D} K(\cdot,\zeta)\nu_n(d\zeta) = \lim_n u_n.$$

Since $\mathcal{H}_{a,x}$ is closed for pointwise convergence, $v \in \mathcal{H}_{a,x}$.

We claim that ν is concentrated on $\{z\}$.

For $n \geq 2$, let f be a continuous function on ∂D such that $f \equiv 1$ on $\partial D \setminus \Delta_{n-1}$, $f \equiv 0$ on $\mathrm{Cl}\Delta_{n-1} \subset \Delta_n$ and $0 \leq f \leq 1$. Then, for every $m \geq n$, $f \equiv 0$ on Δ_m, and, therefore, $\int_{\partial D} f(\zeta)\nu_m(d\zeta) = 0$. But

$$\int_{\partial D} f(\zeta)\nu_m(d\zeta) \to \int_{\partial D} f(\zeta)\nu(d\zeta).$$

Hence,

$$0 = \int_{\partial D} f(\zeta)\nu(d\zeta) \geq \int_{\partial D\setminus\Delta_{n-1}} f(\zeta)\nu(d\zeta) = \nu(\partial D \setminus \Delta_{n-1}).$$

Thus,

$$\nu(\partial D \setminus \{z\}) = \nu(\cup_{n=1}^{\infty}(\partial D \setminus \Delta_n)) = 0,$$

so ν sits at $\{z\}$.

Therefore, $\nu = c\epsilon_z$ for some positive constant c. Further

$$1 = u_n(a) = \int_{\partial D} K(a,\zeta)\nu_n(d\zeta) \to \int_{\partial D} K(a,\zeta)\nu(d\zeta) = cK(a,z).$$

Hence, $v = \dfrac{K(\cdot,z)}{K(a,z)} \in \mathcal{H}_{a,x}$. □

It is evident that the kernel function K on $D \times \partial D$ has the central role in all considerations about the cone $\mathcal{H}$. On the other hand, we saw in Section 2.1 that the function $k(x,y) = \inf\{u(x)/u(y) : U \in \mathcal{H}\}$ (denoted by $s_y(x)$ here) had distinguished place among the functions in $\mathcal{H}^{\inf}$. It would be nice if the kernel function K could be recovered from the function k. This is true with the additional hypothesis on K:

(H_3) For all $z_1, z_2 \in \partial D$ such that $z_1 \neq z_2$,

$$\lim_{(x,z)\to(z_1,z_2)} K(x,z) = 0,$$

where $(x,z) \in D \times \partial D$ and $(x,z) \to (z_1,z_2)$ in $\overline{D} \times \overline{D}$.

This hypothesis is in some sense stronger than the previous two and will restrict the examples in the next chapter to nice domains. Still, as we shall see, there are many interesting examples satisfying (H_3).

Let us fix $x \in D$ (x_0 is still fixed). For each $a \in D$, there exists $z = z(a,x) \in \partial D$ such that $K(\cdot,z)/K(a,z) \in \mathcal{H}_{a,x}$. Denote $z(a,x)$ by z_a.

Recall that we have assumed the existence of the function $u_0 \in \mathcal{H}$ satisfying $m \leq u_0 \leq M$. For any a in D, $u_0(x)/u_0(a) \leq M/m$. Therefore, $s_a(x) \leq M/m$.

<u>LEMMA 2.10</u> *Let $a \to z$, $z \in \partial D$. Then $z_a \to z$.*

PROOF: Let $\{a_n\}$ be a sequence in D converging to z and let us denote the corresponding points on the boundary by $\{z_n\}$. Since ∂D is compact, we may assume that $\{z_n\}$ converges to some point $z_0 \in \partial D$. If $z_0 \neq z$, then by (H_3), $\lim_n K(a_n, z_n) = 0$. By the continuity of K, $\lim_n K(x,z_n) = K(x,z_0) < \infty$. Therefore, the sequence $\{K(x,z_n)/K(a_n,z_n)\}$ is unbounded. On the other hand,

$$\frac{K(x,z_n)}{K(a_n,z_n)} = s_{a_n}(x) \leq \frac{M}{m}.$$

Contradiction! Hence $z_0 = z$. □

<u>THEOREM 2.2</u> *For every $z \in \partial D$,*

$$\lim_{a\to z} \frac{s_a}{s_a(x_0)} = K(\cdot, z).$$

PROOF: Recall that $x_0 \in D$ such that $K(x_0, \cdot) \equiv 1$ on ∂D. Fix $x \in D$. Let v_a and z_a be points on ∂D such that $K(\cdot, v_a)/K(a, v_a) \in \mathcal{H}_{a,x_0}$ and $K(\cdot, z_a)/K(a, z_a) \in \mathcal{H}_{a,x}$, $a \in D$. Thus,

$$\frac{K(x, z_a)}{K(a, z_a)} = s_a(x) \text{ and } \frac{K(x_0, v_a)}{K(a, v_a)} = s_a(x_0). \tag{2.15}$$

By definition of s_a,

$$\frac{K(x, v_a)}{K(a, v_a)} \geq s_a(x) \text{ and } \frac{K(x_0, z_a)}{K(a, z_a)} \geq s_a(x_0). \tag{2.16}$$

From (2.15) and (2.16) it follows

$$\frac{K(a, z_a)}{K(a, v_a)} \geq \frac{K(x, z_a)}{K(x, v_a)} \text{ and } \frac{K(a, z_a)}{K(a, v_a)} \leq 1. \tag{2.17}$$

As $a \to z$, Lemma 2.10 gives that $z_a \to z$ and $v_a \to z$. The first inequality above and continuity of K give

$$\liminf_{a \to z} \frac{K(a, z_a)}{K(a, v_a)} \geq \liminf_{a \to z} \frac{K(x, z_a)}{K(x, v_a)} = \frac{K(x, z)}{K(x, z)} = 1.$$

From the second inequality in (2.17) it follows that

$$\limsup_{a \to z} \frac{K(a, z_a)}{K(a, v_a)} \leq 1.$$

Hence,

$$\lim_{a \to z} \frac{K(a, z_a)}{K(a, v_a)} = 1. \tag{2.18}$$

By (2.15),

$$\frac{s_a(x)}{s_a(x_0)} = \left[\frac{K(x, z_a)}{K(a, z_a)}\right] \left[\frac{K(x_0, v_a)}{K(a, v_a)}\right]^{-1} = K(x, z_a) \frac{K(a, v_a)}{K(a, z_a)}.$$

Therefore, by (2.18) and continuity of K,

$$\lim_{a \to z} \frac{s_a(x)}{s_a(x_0)} = \lim_{a \to z} K(x, z_a) \frac{K(a, v_a)}{K(a, z_a)} = K(x, z) .$$

□

CHAPTER 3
EXAMPLES OF CONES WITH COMPACT BASIS

3.1 Positive Solutions of Elliptic Differential Equations

Let D be a bounded domain in $\mathbf{R}^n$, and let

$$A = \sum_{i,j=1}^{n} D_i(a_{ij}(x)D_j)$$

be an elliptic operator in divergence form with bounded coefficients a_{ij} satisfying $a_{ij} = a_{ji}$. We shall assume that A is uniformly elliptic in D, i.e., there exist constants λ and Λ, $0 < \lambda < \Lambda$, such that

$$\lambda|\xi|^2 \leq \sum a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2$$

for all $\xi \in \mathbf{R}^n$ and $x \in D$.

A weak solution of the equation $Au = 0$ in D is a function u in the Sobolev space $H^{1,2}_{\text{loc}}(D) = \{u \in L^2_{\text{loc}}(D) : \nabla u \in L^2_{\text{loc}}(D)\}$ satisfying

$$\sum_{i,j=1}^{n} \int_D a_{ij}(x)D_iu(x)D_j\phi(x)\ dx\ = 0 \tag{3.1}$$

for every $\phi \in \mathcal{C}_c^\infty(D)$.

It is well-known that every solution of $Au = 0$ is locally Hölder continuous in D: For each compact subset K of D

$$\sup_{x,y\in K} \frac{|u(x) - u(y)|}{|x-y|^\alpha} < \infty, \quad \text{where } 0 < \alpha \leq 1.$$

This is a famous result discovered by DeGiorgi and Nash. See [13, p. 190] for a proof.

In particular, local Hölder continuity of solutions has as a consequence the fact that solutions are locally uniformly equicontinuous (and, of course, continuous on D).

The next result we need is Harnack inequality proved by Moser. A proof can be found in [13]. A more convenient form is the one given in [4]. Let K be a compact subset of D. Then there exists a constant $c = c(\lambda, \Lambda, K, D, n)$ such that for every nonnegative solution of $Au = 0$,

$$\max_K u \leq c \min_K u.$$

By linearity of A, nonnegative solutions of $Au = 0$ form a convex cone. This cone is the object of our interest. Formally, let

$$\mathcal{H} = \{u : Au = 0, u \geq 0\}. \tag{3.2}$$

All functions in $\mathcal{H}$ are continuous. If $u(x) = 0$ for some x in D, $u \in \mathcal{H}$, then Harnack inequality immediately implies that u is identically zero. Thus, $\mathcal{H}$ consists of strictly positive functions (unless zero).

Let us show that $\mathcal{H}$ is closed in $\mathcal{C}(D)$. The following estimate is needed.

LEMMA 3.1 *Let $y \in D$ and $0 < r < R$ such that the ball $B(y, R)$ is contained in D. For any solution u of $Au = 0$ the inequality*

$$\int_{B(y,r)} |\nabla u(x)|^2 \, dx \leq C \int_{B(y,R)} u^2(x) \, dx \tag{3.3}$$

holds, where the constant C depends on λ, Λ, r and R.

This lemma is standard and is stated in a similar form in [18]. For a proof, first note that (3.1) is also valid for $\phi \in H_0^{1,2}(D)$. Let ψ be $\mathcal{C}_c^\infty$ function identically 1 in $B(y, r)$, zero outside $B(y, R)$ and with bounded gradient. Let $\phi = u\psi^2$; then $\phi \in H_0^{1,2}$. Equation (3.1) can be written as

$$-\int_{B(y,R)} \psi^2 \sum_{i,j} a_{ij} D_i u D_j u \, dx = 2\int_{B(y,R)} u\psi \sum_{i,j} D_i u D_j \psi \, dx.$$

Repeated use of ellipticity and Schwartz inequality proves the lemma.

Now let $\{u_k\}$ be a sequence in $\mathcal{H}$ and assume that $u_k \to u$ in $\mathcal{C}(D)$. First we show that u is in $H^{1,2}_{\text{loc}}(D)$. Let $y \in D$, $0 < r < R$ with $B(y,R) \subset D$. Since $u_k - u_m$ is a solution of $Au = 0$, the lemma above yields

$$\int_{B(y,r)} |\nabla(u_k - u_m)|^2 \, dx \leq C \int_{B(y,R)} (u_k - u_m)^2 \, dx.$$

Since $u_k \to u$ uniformly in $B(y,R)$, the integral on the right hand side is arbitrarily small for large k and m. Hence, the left-hand side is also small. Therefore, for $\epsilon > 0$, there is $k_0 \in \mathbf{N}$ such that for $k, m \geq k_0$

$$\int_{B(y,r)} (|u_k - u_m|^2 + |\nabla(u_k - u_m)|^2) dx = \| u_k - u_m \|^2_{H^{1,2}(B(y,r))} < \epsilon.$$

Here $H^{1,2}(B(y,r))$ denotes Sobolev space on $B(y,r)$. By completeness, there exists $v \in H^{1,2}(B(y,r))$ such that $u_k \to v$ in $H^{1,2}(B(y,r))$. In particular, $u_k \to v$ in $L^2(B(y,r))$. But $u_k \to u$ uniformly on $B(y,r)$, so $u = v$ a.e. in $B(y,r)$. Therefore, $u \in H^{1,2}(B(y,r))$. Since y and r were arbitrary, $u \in H^{1,2}_{\text{loc}}(D)$.

Furthermore, $u_k \to u$ in $H^{1,2}(B(y,r))$, which implies that $u_k \to u$ in $H^{1,2}(D_1)$ for every relatively compact open subset D_1 of D.

If $\phi \in \mathcal{C}^\infty_c(D)$ with $\text{Supp}(\phi) \subset D_1$, then (3.1) holds for every u_k. Note that $\max_j \| D_j\phi \|_\infty < \infty$ and $\max_{i,j} \| a_{ij} \|_\infty < \infty$. Let c be a common bound for both maxima. Then

$$\left| \int_D \left(\sum_{i,j} a_{ij}(x)(D_i u_k - D_i u) D_j \phi \right) dx \right|$$

$$\leq \int_{D_1} \left(\sum_{i,j} \| a_{ij} \|_\infty |D_i u_k - D_i u| \, \| D_j \phi \|_\infty \right) dx \leq c^2 \int_{D_1} \left(\sum_{i,j} |D_i u_k - D_i u| \right) dx$$

$$= nc^2 \int_{D_1} \left(\sum_i |D_i u_k - D_i u| \right) dx \le nc^2 \sum_i m(D_1)^{1/2} \left(\int_{D_1} |D_i(u_k - u)|^2 dx \right)^{1/2}$$

$$= nc^2 \, m(D_1)^{1/2} \sum_i \left(\int_{D_1} |D_i(u_k - u)|^2 dx \right)^{1/2},$$

which tends to zero since $u_k \to u$ in $H^{1,2}(D_1)$. Thus

$$\left| \sum_{i,j} \int_D a_{ij}(x) D_i u D_j \phi \, dx \right| = \left| \sum_{i,j} \int_D a_{ij}(x)(D_i u - D_i u_k) D_j \phi \, dx \right|$$

is arbitrarily small for large k, hence zero. Since ϕ was an arbitrary function in $C_c^\infty(D)$, it follows that $Au = 0$. Therefore, $u \in \mathcal{H}$; so $\mathcal{H}$ is closed.

Let x_0 be an arbitrary, but fixed, point in D and let

$$\mathcal{H}_{x_0} = \{u : u \in \mathcal{H}, u(x_0) = 1\}. \tag{3.4}$$

Then $\mathcal{H}_{x_0}$ is closed in $\mathcal{C}(D)$, locally uniformly equicontinuous and bounded (by Harnack inequality). Further, $\mathcal{H}_{x_0}$ is obviously a basis for $\mathcal{H}$. Hence, $\mathcal{H}$ has a compact basis. Note that $\mathcal{H}$ contains positive constants. This shows that $\mathcal{H}$ is a cone of the type studied in Chapter 2, so all results from Section 2.1 are applicable. In particular, the cone $\mathcal{H}^{\inf}$ of all possible infima of positive solutions of $Au = 0$ consists of continuous functions and is closed for pointwise convergence. Of course, the case of the Laplacian $\Delta = \sum \partial^2/\partial x_i{}^2$ is also included.

In order to apply results of Section 2.2, some restrictions on the domain D are needed. We shall assume that D is a bounded Lipschitz domain in $\mathbf{R}^n$.

A bounded domain $D \subset \mathbf{R}^n$ is called a *Lipschitz domain* if there are positive numbers r_0 and m such that for every point $z \in \partial D$

$$B(z, r_0) \cap D = B(z, r_0) \cap \{(x, t) : x \in \mathbf{R}^{n-1}, t > \phi(x)\}, \quad \text{and}$$

$$B(z, r_0) \cap \partial D = B(z, r_0) \cap \{(x, \phi(x)) : x \in \mathbf{R}^{n-1}\},$$

where ϕ is a function satisfying $\| \nabla \phi \|_{L^\infty(\mathbf{R}^{n-1})} \le m < \infty$.

We denote $B(z,r) \cap \partial D$ by $\Delta(z,r)$. For $s > 1$, let

$$\psi(z,r,sr) = \{(y,t) : |y - y_0| < r, |t - \phi(y_0)| < sr\}, \quad z = (y_0, \phi(y_0)),$$

be a cylinder, and let $v_r = v_r(z) = (y_0, \phi(y_0) + r)$ be the "center" of that cylinder.

Let us recall the definition of the elliptic-harmonic measure. The Dirichlet problem for A in D is solvable if for every continuous function g on ∂D, there is a unique continuous function u on $\bar{D}$ such that $Au = 0$ in D and $u_{|\partial D} = g$. A result from [18] states that the Dirichlet problem for A in D is solvable if and only if it is solvable for the Laplacian Δ. Since D is Lipschitz, it follows that the Dirichlet problem for A in D is solvable.

For $x \in D$, the mapping $g \mapsto u(x)$ is a positive linear functional on $\mathcal{C}(\partial D)$. Hence, there is a positive measure ω^x on ∂D such that

$$u(x) = \int_{\partial D} g(\zeta)\omega^x(d\zeta).$$

It is easy to see that ω^x is a probability measure. We call ω^x the elliptic-harmonic measure for A in D at x.

Fix $z \in \partial D$. A function $K(\cdot, z)$ defined on D is called a *kernel function at z* (for the operator A and domain D) if

(i) $K(\cdot, z)$ is a solution of $Au = 0$ in D,

(ii) $K(\cdot, z) \in \mathcal{C}(\bar{D} \setminus \{z\})$ and $\lim_{x \to \zeta} K(x,z) = 0$ for $\zeta \in \partial D,\ \zeta \neq z$, and

(iii) $K(x,z) > 0$ for each $x \in D$ and $K(x_0, z) = 1$, where x_0 is the fixed point in D.

The existence of the kernel function for A is established in [4, Theorem 3.1] with the additional property that the function $z \mapsto K(x,z)$ is continuous on the boundary ∂D ([4, Corollary 3.2]). Therefore, K, considered as a function of two variables on $D \times \partial D$, satisfies the hypothesis (H_1) from Section 2.2.

The representation theorem was also obtained in [4, Theorem 4.1] (see also [9]): For each $u \in \mathcal{H}$, there is a unique Borel measure μ on ∂D such that

$$u(x) = \int_{\partial D} K(x,z)\mu(dz).$$

Thus, our hypothesis (H_2) also holds.

To establish (H_3) we need to show that whenever $(x,z) \in D \times \partial D$ converges to (z_1, z_2), $z_1, z_2 \in \partial D$, $z_1 \neq z_2$, the kernel function $K(x,z)$ tends to zero. Once again we reach for a result from [4]. This time it is their Lemma 2.5

<u>LEMMA 3.2</u> *Suppose $r \leq r_0/2$ and u is a positive solution of $Au = 0$ with*

(i) $u \in \mathcal{C}(\overline{D \setminus \psi(z_2, r, sr)})$, *and*

(ii) $u = 0$ *on* $\partial D \setminus \overline{\Delta(z_2, r)}$.

Then, for $x \in D \setminus \psi(z_2, 2r, 2sr)$,

$$c^{-1}u(v_r)\omega^x(\Delta(z_2,r)) \leq u(x) \leq cu(v_r)\omega^x(\Delta(z_2,r))$$

(where $v_r = v_r(z_2)$) with c depending only on λ, Λ and M.

We use this lemma for functions $x \mapsto K(x,z)$. For $z_1 \neq z_2$, there exists $r > 0$ such that $z_1 \notin \psi(z_2, 3r, 3sr)$. If $z \in \psi(z_2, r, sr)$, then $K(\cdot, z)$ satisfies assumptions (i) and (ii). Hence, for $x \in \psi(z_2, 2r, 2sr)$,

$$K(x,z) \leq cK(v_r, z)\omega^x(\Delta(z_2, r)).$$

As $x \to z_1$, $\omega^x(\Delta(z_2,r)) \to 0$, and as $z \to z_2$, $K(v_r, z) \to K(v_r, z_2) < \infty$. Hence, $K(x,z) \to 0$ as $(x,z) \to (z_1, z_2)$. This proves (H_3).

3.2 Positive Solutions of Schrödinger Equation

In this section we consider positive solutions of the Schrödinger equation. The notation remains as in Section 3.1: D is a bounded domain in $\mathbf{R}^n$, but here we require that $n \geq 3$.

Let $L = -A + q$ be the Schrödinger operator on D. A is a uniformly elliptic operator in divergence form as in 3.1 with bounds λ and Λ, and q is a function in the Kato class $K_n(D)$, i.e.,

$$\lim_{r \to 0} \sup_{x \in D} \int_{|x-y| \leq r} \frac{|q|(y)}{|x-y|^{n-2}} dy = 0. \tag{3.5}$$

A weak solution of $Lu = 0$ is a function u in the Sobolev space $H^{1,2}_{\text{loc}}(D)$ satisfying

$$-\sum_{i,j=1}^{n} \int_D a_{ij}(x) D_i u(x) D_j \phi(x)\, dx = \int_D q(x) u(x) \phi(x)\, dx \tag{3.6}$$

for every function $\phi \in C_c^\infty(D)$.

Continuity of solutions and Harnack inequality for nonnegative solutions was established in [5]. They proved the following two theorems.

Continuity Theorem: *There exists a nondecreasing function $\omega(s)$ depending on λ and Λ such that $\omega(s) \to 0$ as $s \to 0$ and, for any solution of $Lu = 0$ in D, and for any ball $B_r(y)$ with $B_{3r}(y) \subset D$,*

$$|u(x) - u(y)| \leq \omega\left(\frac{|x-y|}{r}\right) \sup_{B_{3r}(y)} |u|, \quad x \in B_r(y). \tag{3.7}$$

Harnack's Theorem: *There exist positive constants r_0 and C depending on λ, Λ and n, such that for any nonnegative solution u of $Lu = 0$ in D and, for any ball B_r with $0 < r \leq r_0$ and $B_{2r} \subset D$, we have*

$$\sup_{B_{r/2}} u \;\leq\; C \inf_{B_{r/2}} u.$$

From Harnack's Theorem it follows that for every compact subset K of D there is a constant $C = C(\lambda, \Lambda, n, K)$ such that for any nonnegative solution u of $Lu = 0$ in D we have

$$\sup_K u \leq C \inf_K u. \tag{3.8}$$

This follows from the familiar chain argument.

Let $\mathcal{H}$ denote the family of all nonnegative solutions of $Lu = 0$ in D. Note that Harnack inequality implies that every nonnegative solution which is not identically zero is necessarily strictly positive. Let x_0 be an arbitrary, fixed point in D and let

$$\mathcal{H}_{x_0} = \{u : u \in \mathcal{H}, u(x_0) = 1\}. \tag{3.9}$$

Harnack inequality immediately shows that $\mathcal{H}_{x_0}$ is bounded in $\mathcal{C}(D)$. Let y be any point in D, and choose $r > 0$ such that $B_{3r}(y) \subset D$. Then $\mathcal{H}_{x_0}$ is bounded on $B_{3r}(y)$. Now (3.7) implies the uniform equicontinuity of $\mathcal{H}_{x_0}$ at y. Since this holds for every $y \in D$, $\mathcal{H}_{x_0}$ is locally uniformly equicontinuous.

It remains to show that $\mathcal{H}$ is closed in $\mathcal{C}(D)$. This is proved in the exactly same way as in Section 3.1. Lemma 3.1 is valid for solutions of $Lu = 0$, and the proof can be found in [5].

Hence, $\mathcal{H}$ is a closed convex cone with compact basis $\mathcal{H}_{x_0}$. Thus, we can form the cone $\mathcal{H}^{\text{inf}}$ consisting of all possible infima of positive solutions of $Lu = 0$. Functions in $\mathcal{H}^{\text{inf}}$ are continuous, and $\mathcal{H}^{\text{inf}}$ is closed for pointwise convergence. So again, everything worked fine. But, contrary to the case of the operator A, this time $\mathcal{H}$ may consist of the zero function only.

In order to obtain non-void results, both the domain D and the function q need to be specialized. As in 3.1, we suppose that D is a bounded Lipschitz domain. The function q must have finite gauge. To see what that means we follow [9]. We will

only outline the set-up. The detailed discussion can be found in the aforementioned paper.

Let dx denote the Lebesgue measure on D. The symmetric form $\mathcal{E}$ on $L^2(D, dx)$ associated with A is defined by

$$\mathcal{E}(u,v) = \sum_{i,j=1}^{n} \int_D a_{ij}(x) D_i u(x) D_j v(x)\, dx, \qquad \mathcal{D}[\mathcal{E}] = \mathcal{C}_0^\infty(D). \tag{3.10}$$

This form is *Markovian*, meaning that the following axiom is satisfied:

For each $\epsilon > 0$ there exists a real function $\phi_\epsilon(t)$, $t \in \mathbf{R}$, such that

$$\phi_\epsilon(t) = t \quad \forall t \in [0,1], \quad -\epsilon \le \phi_\epsilon(t) \le 1 + \epsilon \quad \forall t \in \mathbf{R}, \tag{3.11}$$

$$\text{and } 0 \le \phi_\epsilon(t') - \phi_\epsilon(t) \le t' - t \ \text{ whenever } t \le t',$$

$$u \in \mathcal{D}[\mathcal{E}] \Rightarrow \phi_\epsilon(u) \in \mathcal{D}[\mathcal{E}] \text{ and } \mathcal{E}(\phi_\epsilon(u), \phi_\epsilon(u)) \le \mathcal{E}(u,u) \tag{3.12}$$

(see [12, p.7]).

The form (3.10) has the *local property*, i.e., if $u, v \in \mathcal{D}[\mathcal{E}]$, and $\mathrm{Supp}(u)$ and $\mathrm{Supp}(v)$ are disjoint compact sets, then $\mathcal{E}(u,v) = 0$.

The form (3.10) is not closed, but it is closable ([12, p.43]). Its smallest extension $\bar{\mathcal{E}}$ is a *regular Dirichlet form* possessing the local property ([12, pp.41,42]). This means that $\bar{\mathcal{E}}$ is a closed, symmetric Markovian form possessing a core. A *core* of $\bar{\mathcal{E}}$ is by definition a subset C of $\mathcal{D}[\bar{\mathcal{E}}] \cap \mathcal{C}_0(D)$ such that C is dense in $\mathcal{D}[\bar{\mathcal{E}}]$ with respect to the norm $\|u\| = \mathcal{E}(u,u) + (u,u)$, and C is dense in $\mathcal{C}_0(D)$ in the uniform norm. By Theorem 6.2.2 in [12], there exists a diffusion (X_t, P^x) on D, with no killing inside D, whose Dirichlet form is the one given by $\mathcal{E}$.

We assume that (X_t, P^x) is defined on the canonical space of paths $(\Omega, \mathcal{F}_t, \theta_t)$. Here

$$\Omega = \{\omega : t \mapsto \omega(t) \text{ is continuous on } (0, \zeta(\omega)),\ \omega(t) = \partial,\ t \ge \zeta(\omega)\},$$

where $\zeta(\omega)$ is the lifetime of ω and ∂ denote the "cemetery". The filtration $(\mathcal{F}_t)$ is natural and θ_t is the shift.

We also need the conditional diffusion starting at x and dying at y. There is a measure P_y^x on Ω such that $P_y^x(X_{0+} = x) = 1$ and $P_y^x(X_{\tau_D -} = y) = 1$ where τ_D is the killing time of X (i.e., the exit time from D). For $x \in D$, such measure is constructed by using Doob's h-transform, while for $x \in \partial D$, time-reversal is also used.

For the function q, let

$$e_q(t) = \exp\{-\int_0^t q(X_s)ds\}. \tag{3.13}$$

The *gauge* of (A, D, q) is defined by

$$F(x) = E^x[e_q(\tau_D)], \quad x \in D, \tag{3.14}$$

and the *conditional gauge* by

$$F(x,y) = E_y^x[e_q(\tau_D)], \quad x, y \in \overline{D}. \tag{3.15}$$

Here τ_D is the exit time from D, i.e., the lifetime. If the gauge F is not identically infinite, then it is bounded. Moreover, finiteness of the gauge is equivalent to either of the following two conditions:

(i) There is a solution in D to $Lu = 0$ with $\inf_D u > 0$.
(ii) For each $f \in \mathcal{C}(\partial D)$ with $f \geq 0$, there is a nonnegative solution to the Dirichlet problem

$$Lu(x) = 0, \ x \in D, \qquad u(z) = f(z), \ z \in \partial D$$

(for the proof see [9]).

From now on we assume that $F \not\equiv \infty$. Then the cone $\mathcal{H}$ is nontrivial. Furthermore, from (ii) it follows that there is a function in $\mathcal{H}$ which is bounded from below

(by a strictly positive constant) and from above. Hence, $\mathcal{H}$ satisfies the assumptions from 2.2. It remains to check hypotheses (H_1)-(H_3).

In [9] it is proved that the conditional gauge $F(\cdot,\cdot)$ is continuous on $\overline{D}\times\overline{D}$ and that there exist constants c_1 and c_2 , $0 < c_1 < c_2 < \infty$, such that $c_1 \le F(x,y) \le c_2$, for all $(x,y) \in \overline{D}\times\overline{D}$ (see their theorems 4.8 and 4.6). By Theorem 5.5 in [9], every function $u \in \mathcal{H}$ has the unique representation

$$u(x) = \int_{\partial D} K_L(x,z)\mu(dx) \tag{3.16}$$

for some Borel measure μ on ∂D, where

$$K_L(x,z) = \frac{F(x,z)}{F(x_0,z)} K(x,z) \tag{3.17}$$

(recall that x_0 is fixed in D and K is the kernel function for A). Continuity of K_L in both variables follows from (3.17) and the continuity of K and F. Finally,

$$\lim_{(x,z)\to(z_1,z_2)} K_L(x,z) \le \frac{c_2}{c_1} \lim_{(x,z)\to(z_1,z_2)} K(x,z) = 0,$$

for all $z_1, z_2 \in \partial D,\ z_1 \ne z_2$. Hence, all three hypotheses are satisfied.

3.3 Harmonic Functions for a Diffusion

In this section we switch from the analytic setting to a probabilistic one and describe the cone of harmonic functions for a certain diffusion. We adopt the notation from [3] as is customary in Markov process theory. So, the state space is still locally compact with countable basis, but now it is denoted by E. In addition we assume that E is connected. Let $\mathcal{E}$ be the Borel σ-algebra on E.

We assume that $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x)$ is a transient diffusion on the state space $(E,\mathcal{E})$. By this we mean that X is a transient Hunt process with continuous paths. Let T_A and τ_A denote the hitting time of a set A, and the exit time from A,

respectively, i.e.,

$$T_A = \inf\{t > 0 : X_t \in A\} \quad \text{and} \quad \tau_A = \inf\{t > 0 : X_t \notin A\}.$$

Transience means that for every compact subset K of E, and for every x,

$$\lim_{t \to \infty} P^x(T_K \circ \theta_t) = 0.$$

Let (P_t) denote the transition semigroup of X and let U be the potential operator defined by

$$Uf(x) = \int_0^\infty P_t f(x)\, dt$$

for f nonnegative Borel measurable function. Following [8] and [21] we assume that there is a reference measure ξ on $(E, \mathcal{E})$ such that the potential kernel $U(x, dy)$ has the density $u(x, y)$ with respect to ξ. That is

$$U(x, dy) = u(x, y)\xi(dy).$$

It is assumed that $u(x, y)$ satisfies the following two properties:

(i) $(x, y) \mapsto u^{-1}(x, y)$ is finite and continuous, and

(ii) $u(x, y) = \infty$ if and only if $x = y$.

Note that (i) implies that $u(x, y) > 0$.

A nonnegative measurable function f is *excessive* if (i) $P_t f \leq f$ for every $t > 0$, and (ii) $\lim_{t \to 0} P_t f = f$.

For a Borel set A and a measurable function f, let

$$P_A f(x) = E^x[f(X_{T_A}); T_A < \infty]$$

be the hitting operator of A. A nonnegative measurable function h is called *harmonic* (for X) in an open subset D of E, if

$$h(x) = P_{K^c} h(x) \tag{3.18}$$

for every compact subset K of D. If $D = E$, then we say that h is harmonic in E.

It was proved in [8, p.127, Thm.6], that every excessive function s which is not identically infinite has the Riesz decomposition

$$s = h + \int_E u(x,y)\mu(dy) \tag{3.19}$$

where h is excessive and harmonic, and μ is a Radon measure on $(E, \mathcal{E})$. Furthermore, s is harmonic in D if and only if $\mu(D) = 0$ (cf. [21, p.610, Thm.2].

The result which makes harmonic function tractable in view of Chapter 2 is the following one [21, p.610, Thm.3].

<u>THEOREM 3.1</u> *Let h be an excessive function, $h \not\equiv \infty$. If h is harmonic in an open set D, then h is continuous in D.*

We outline the proof following [21].

PROOF: We assume that D is relatively compact open. The function $s = P_D h$ is excessive. Since $s = h$ in D, the continuity of paths of X implies that s is harmonic in D. Let $s = U\mu$: then μ is carried by $\overline{D}$ and since s is harmonic in D, $\mu(D) = 0$. Further,

$$s(x) = P_D h(x) = \int_{\partial D} u(x,y)\mu(dy). \tag{3.20}$$

Continuity assumption for u yields that the right-hand side is continuous. Hence, h is continuous in D.□

With the same assumptions as in [21], it is proved in [22, Cor.1], that every nonnegative harmonic function for X is automatically excessive. So, the theorem above says that all nonnegative harmonic functions are continuous. Let $\mathcal{H}$ denote the family of all nonnegative functions which are harmonic for X on the whole space E. Then $\mathcal{H}$ is a convex cone of nonnegative continuous functions. Every function

in $\mathcal{H}$ is strictly positive (unless identically zero). Indeed, let $h \in \mathcal{H}$ and $h(x_0) = 0$ for some $x_0 \in E$. Let D be a relatively compact open subset of E containing x_0. If $s = P_D h$, then $s = \int_{\partial D} u(x,y)\mu(dy)$ for a Radon measure μ concentrated on ∂D. Further, $s = h$ in D, hence $0 = s(x_0) = \int_{\partial D} u(x_0,y)\mu(dy)$. But $u(x_0,y) > 0$ implies that the measure μ is zero. Therefore $s = 0$ and, in particular, $h = 0$ in D. Since E can be covered by a sequence of relative compact open subsets of E, it follows that $h = 0$ in E.

Following the pattern from sections 3.1 and 3.2, we prove that $\mathcal{H}$ is closed in $\mathcal{C}(E)$.

<u>PROPOSITION 3.1</u> *The cone $\mathcal{H}$ is closed in $\mathcal{C}(E)$.*

PROOF: Let $\{h_n\}$ be a sequence of functions in $\mathcal{H}$ which converges to a function h uniformly on compacts. Let K be a compact subset of E. If $x \in K$ is regular for K^c, i.e., if $P^x(T_{K^c} = 0) = 1$, then $X_{T_{K^c}} = x \quad P^x$a.e., so $P_{K^c}h(x) = h(x)$. Note that since K^c is open, every point of K^c is regular for K^c. Therefore irregular points for K^c are only in K.

Let x be irregular for K^c, i.e., $T_{K^c} > 0 \quad P^x$a.e. Then $x \in K$ and by continuity of X we have $X_{T_{K^c}} \in \partial K, \;\; P^x$a.e. For $\epsilon > 0$, we can find n_0 large enough such that $|h_n(y) - h(y)| < \epsilon$ for all $y \in K$ and all $n \geq n_0$. Further,

$$|P_{K^c}h_n(x) - P_{K^c}h(x)| \leq E^x[|h_n(X_{T_{K^c}}) - h_n(X_{T_{K^c}})|; T_{K^c} < \infty] < \epsilon P^x[T_{K^c} < \infty] \leq \epsilon,$$

for $n \geq n_0$. Therefore,

$$|h(x) - P_{K^c}h(x)| \leq |h(x) - h_n(x)| + |h_n(x) - P_{K^c}h_n(x)| + |P_{K^c}h_n(x) - P_{K^c}h(x)| < 2\epsilon,$$

since the middle term is zero by assumption. This proves that h is harmonic.□

Now we fix a point x_0 in E and consider the family

$$\mathcal{H}_{x_0} = \{h : h \in \mathcal{H}, \;\; h(x_0) = 1\}. \tag{3.21}$$

Then $\mathcal{H}_{x_0}$ is obviously a basis for the cone $\mathcal{H}$. Our next goal is to show that it is compact in $\mathcal{C}(E)$. This follows from Theorems 4 and 5 in [21]. Theorem 4 shows that Harnack inequality holds, while Theorem 5 gives the local equicontinuity of $\mathcal{H}_{x_0}$. We take another approach and prove directly that $\mathcal{H}_{x_0}$ is compact without invoking Harnack inequality.

PROPOSITION 3.2 *$\mathcal{H}_{x_0}$ is compact in $\mathcal{C}(E)$.*

PROOF: Let $\{h_n\}$ be a sequence in $\mathcal{H}_{x_0}$. We want to show that there is a subsequence $\{h_{n_k}\}$ and h in $\mathcal{H}_{x_0}$ such that $h_{n_k} \to h$ in $\mathcal{C}(E)$.

Let K be a compact subset of E and, without loss of generality, we assume that $x_0 \in K$. Let D be a relatively compact open set in E containing K. For each $n \in \mathbf{N}$ we define $v_n = P_D h_n$. Then by (3.20)

$$v_n(x) = \int_{\partial D} u(x,y)\mu_n(dy)$$

where μ_n is a positive measure on ∂D. Further, $P_D h_n = h_n$ in D.

Since $y \mapsto u(x_0, y)$ is continuous and positive on ∂D, it attains its minimum, say $m > 0$. Then

$$1 = h_n(x_0) = P_D h_n(x_0) = \int_{\partial D} u(x_0,y)\mu_n(dy) \geq \int_{\partial D} m\mu_n(dy).$$

Hence, $\mu_n(dy) \leq 1/m$ for every n, and thus $\{\mu_n\}$ is a bounded sequence of measures. Let $\{\mu_{n_k}\}$ be its weakly convergent subsequence and μ the limit measure, which is also concentrated on ∂D.

Let $v = U\mu = \int_{\partial D} u(\cdot, y)\mu(dy)$. Then v is excessive in E and harmonic in D (since $\mu(D) = 0$). For each fixed $x \in D$, the function $y \mapsto u(x,y)$ is continuous on ∂D and therefore

$$\int_{\partial D} u(x,y)\mu_{n_k}(dy) \to \int_{\partial D} u(x,y)\mu(dy),$$

i.e., $h_{n_k}(x) \to v(x)$, for every $x \in D$.

Now we show that this convergence is uniform on K. The function $(x,y) \mapsto u(x,y)$ is uniformly continuous on $K \times \partial D$. Hence, for $\epsilon > 0$, there is $\delta > 0$ such that $|u(x,y) - u(x',y')| < \epsilon$ whenever $d(x,x') + d(y,y') < \delta$ (d is a metric compatible with the topology on E). By compactness of K, there are finitely many balls $B(x_i, \delta)$, $i = 1, \ldots, l$, covering K. Let $k_0 \in \mathbf{N}$ such that for $k \geq k_0$, $|h_{n_k}(x_i) - v(x_i)| < \epsilon$ for $i = 1, \ldots, l$. For a given $x \in K$ we choose the ball $B(x_i, \delta)$ containing x. Then, for $k \geq k_0$,

$$|h_{n_k}(x) - v(x)| = |U\mu_{n_k}(x) - U\mu(x)|$$

$$\leq |U\mu_{n_k}(x) - U\mu_{n_k}(x_i)| + |U\mu_{n_k}(x_i) - U\mu(x_i)| + |U\mu(x_i) - U\mu(x)|$$

$$\leq \int_{\partial D} |u(x,y) - u(x_i,y)| \mu_{n_k}(dy) \ + \ |h_{n_k}(x_i) - v(x_i)| \ + \ \int_{\partial D} |u(x_i,y) - u(x,y)| \mu(dy)$$

$$\leq \epsilon \mu_{n_k}(\partial D) + \epsilon + \epsilon \mu(\partial D) \leq \epsilon(2/m + 1).$$

Therefore, the convergence is uniform on K.

So far it is proved that for every sequence $\{h_n\}$ in $\mathcal{H}_{x_0}$, for every compact set K in E and every relatively compact open set D containing K, there exist a subsequence $\{h_{n_k}\}$ and an excessive function v, harmonic in D, such that $h_{n_k} \to v$ uniformly on K.

Let $\{K_n\}$ be an increasing sequence of compact sets covering E such that $K_n \subset \mathrm{Int}K_{n+1}$ and $x_0 \in K_1$. Let $\{h_n\}$ be a sequence in $\mathcal{H}_{x_0}$. For K_1 there is a subsequence $\{h_{n1}\}$ of $\{h_n\}$ and an excessive function v_1 harmonic in $\mathrm{Int}K_2$, such that $h_{n1} \to v_1$ uniformly on K_1. We proceed inductively: For $j \in \mathbf{N}$ there is a subsequence $\{h_{n,j+1}\}$ of $\{h_{n,j}\}$ and an excessive function v_{j+1}, harmonic in $\mathrm{Int}K_{j+2}$ such that $h_{n,j+1} \to v_{j+1}$ uniformly on K_{j+1}. Obviously, $v_{j+1|K_j} = v_j$.

Let $\{h_{n,n}\}$ be the diagonal subsequence and define the function v on E by $v_{|K_j} = v_j$. Then $h_{n,n} \to v$ uniformly on K_j for every $j \in \mathbf{N}$. This implies that v is

continuous and that $h_{n,n} \to v$ in $\mathcal{C}(E)$. Since every function $h_{n,n}$ is in $\mathcal{H}_{x_0}$, the proof of Proposition 3.1 implies that $v \in \mathcal{H}$. Trivially, $v(x_0) = 1$. □

Together, the last two propositions imply that $\mathcal{H}$ is a closed convex cone with compact basis $\mathcal{H}_{x_0}$. Functions in $\mathcal{H}$ are strictly positive (unless zero). By Corollary 2.1, we see that functions in $\mathcal{H}$ satisfy Harnack inequality. The family $\mathcal{H}^{\text{inf}}$ of all possible infima of functions in $\mathcal{H}$ consists of strictly positive functions. Every function in $\mathcal{H}^{\text{inf}}$ is excessive. Indeed, let $v = \inf_\alpha h_\alpha$ be a function in $\mathcal{H}^{\text{inf}}$ and let D be a relatively compact open subset of E. Then, for every α and every $x \in E$,

$$E^x[v(X_{\tau_D})] \leq E^x[h_\alpha(X_{\tau_D})] = h_\alpha(x)$$

which implies

$$E^x[v(X_{\tau_D})] \leq v(x).$$

Since v is continuous, Theorem 5.11(II) from [3] implies that v is excessive.

By the Riesz representation theorem, (3.19), each $v \in \mathcal{H}^{\text{inf}}$ can be written as $v = h + U\mu$, where h is harmonic and μ is a Radon measure. Obviously, $v - h \in \mathcal{H}^{\text{inf}}$, and hence $U\mu \in \mathcal{H}^{\text{inf}}$.

For the next proposition we need to assume that the state space E satisfies a certain topological property. We note that, for example, star-shaped domains in $\mathbf{R}^n$ have this property.

<u>PROPOSITION 3.3</u> *Let E satisfy the following condition: For every compact subset K of E there exists a relatively compact open set D containing K such that $E \setminus \bar{D}$ is connected. Let $v = U\mu$, $v \neq 0$, be a potential from $\mathcal{H}^{\text{inf}}$. Then μ cannot have compact support.*

PROOF: Assume that Suppμ is compact. Let D be a relatively compact open set containing Suppμ. Then $\mu(E \setminus \bar{D}) = 0$, so v is harmonic in $E \setminus \bar{D}$. Let x_0 be any point

from $E \setminus \bar{D}$. By Lemma 2.7, there is $h \in \mathcal{H}$ such that $h \geq v$ in E and $h(x_0) = v(x_0)$. Then the function $h - v$ is nonnegative and harmonic in $E \setminus \bar{D}$ and $(h-v)(x_0) = 0$. Since $E \setminus \bar{D}$ is connected, the same argument as before shows that $h - v$ must be identically zero on $E \setminus \bar{D}$, i.e., $h = v$ on $E \setminus \bar{D}$. By continuity, $h = v$ on ∂D. Then we have, for any x in D,

$$h(x) = E^x[h(X_{\tau_D})] = E^x[v(X_{\tau_D})] \leq v(x)$$

since $X_{\tau_D} \in \partial D$. Therefore, $h \leq v$ in D. But $h \geq v$ everywhere so, $h = v$ in E. This is a contradiction, because a potential cannot be harmonic. □

If the assumption on E is not satisfied, then it can happen that μ has compact support. To see this, let E be the unit disc in $\mathbf{R}^2$ punctured at the origin: $E = \{x \in \mathbf{R}^2 : 0 < |x| < 1\}$. Let $u = (-\log|x|) \wedge 1$. Since $-\log|x|$ is harmonic and positive in E, $u \in \mathcal{H}^{\inf}(E)$. But the Riesz measure of u sits on the circle centered at the origin with radius $1/e$, so it thus has a compact support.

CHAPTER 4
INFIMA OF HARMONIC FUNCTIONS

4.1 Function k for the Unit Ball

In this chapter we consider in more detail the family of all infima of nonnegative harmonic functions in some domain D in $\mathbf{R}^n$. Here harmonic is understood in the classical sense: $h : D \to \mathbf{R}$ is *harmonic* if $\Delta h = 0$ where Δ denotes the Laplacian. Let $\mathcal{H}$ denote the family of all nonnegative harmonic functions in D, and $\mathcal{H}^{\text{inf}}$ is as usual. Then the discussion in 3.1 shows that $\mathcal{H}$ is a closed convex cone with compact basis, and when D is a bounded Lipschitz domain there is a kernel function satisfying (H_1)-(H_3) and so the results from Chapter 2 are applicable.

In order to compute something, we restrict ourselves to a very special domain — the unit ball in $\mathbf{R}^n$. By using the explicit form of the Poisson kernel and the Möbius transform, it is possible to get some explicit formulae.

Let D be the unit ball in $\mathbf{R}^n$. Recall the definition of functions $s_a,\ a \in D$:

$$s_a(x) = k(x,a) = \inf\{h(x) : h \in \mathcal{H},\ h(a) = 1\}.$$

First we compute s_0 where 0 denotes the origin.

Every positive harmonic function in D satisfies the classical Harnack inequality (e.g. [14, p.31])

$$\frac{1-|x|^2}{(1+|x|)^n}h(0) \le h(x) \le \frac{1-|x|^2}{(1-|x|)^n}h(0). \tag{4.1}$$

If $h(0) = 1$, then the left inequality reads

$$\frac{1-|x|^2}{(1+|x|)^n} \le h(x).$$

Therefore,

$$\frac{1-|x|^2}{(1+|x|)^n} \le s_0(x). \tag{4.2}$$

Let $P(x,z)$, $x \in D$, $z \in \partial D$, denote the Poisson kernel for the unit ball:

$$P(x,z) = \frac{1-|x|^2}{|z-x|^n}. \tag{4.3}$$

For $x \in D$, let z and $-z$ denote points from ∂D which lie on the diameter through x with $-z$ closer to x. Then $|z-x| = 1+|x|$, and so

$$P(x,z) = \frac{1-|x|^2}{(1+|x|)^n}.$$

Together with (4.2) this yields

$$s_0(x) = \frac{1-|x|^2}{(1+|x|)^n}. \tag{4.4}$$

In order to get s_a for arbitrary $a \in D$, we need some results on the Möbius transforms. These can be found in [1] and [17].

A *similarity* in $\mathbf{R}^n$ is the mapping $x \mapsto Mx + b$, where $b \in \mathbf{R}^n$ and M is a *conformal matrix*, i.e., $M = \lambda K$ with $\lambda > 0$ and $K \in O(n)$. Here $O(n)$ denotes the orthogonal group. It is a trivial fact that all similarities preserve harmonicity.

Reflection through the unit sphere is the map $J : x \mapsto x^*$ defined by $x^* = x/|x|^2$. Following [1], let $Q(x)$ denote the matrix with entries $Q(x)_{ij} = x_i x_j/|x|^2$. The derivative of J is

$$J'(x) = \frac{1}{|x|^2}(I - 2Q(x)) \tag{4.5}$$

(see [1, p.18,(14)]), where I denotes the identity matrix. Furthermore, $(I-2Q)^2 = I$, i.e., $I - 2Q \in O(n)$.

The *full Möbius group* $\hat{M}(\mathbf{R}^n)$ is the group generated by all similarities together with J. For any $\gamma \in \hat{M}(\mathbf{R}^n)$, we denote by $|\gamma'(x)|$ the positive number such that

$\gamma'(x)/|\gamma'(x)| \in O(n)$. It follows from (4.5) that $|J'(x)| = 1/|x|^2$. If $\gamma = \lambda K + b$ is a similarity, then $|\gamma'(x)| = \lambda$.

Let h be a positive harmonic function. The *Kelvin transformation* of h is the function

$$\tilde{h}(x) = \frac{1}{|x|^{n-2}} h\left(\frac{x}{|x|^2}\right).$$

Kelvin transformation preserves positivity and harmonicity (e.g., [23, 4.25]). Note that $\tilde{h}$ can be written as

$$\tilde{h} = |J'(x)|^{\frac{n-2}{2}} h(Jx).$$

Hence, if γ is either a similarity or the reflection J, then the map

$$T_\gamma : h \mapsto \tilde{h}, \qquad \tilde{h} = |\gamma'(x)|^{\frac{n-2}{2}} h(\gamma x) \tag{4.6}$$

preserves positivity and harmonicity. By the chain rule, this holds for any $\gamma \in \hat{M}(\mathbf{R}^n)$. Hence the following lemma from [17] holds:

<u>Lemma 4.1</u> *For any $\gamma \in \hat{M}(\mathbf{R}^n)$, the map T_γ defined by (4.6) preserves positivity and harmonicity.*

We shall need a Möbius transformation which maps the unit ball onto itself.

For $a = 0$, let γ_0 be the identity, and for $a \neq 0$,

$$\gamma_a(x) = \frac{(1 - |a|^2)(x - a) - |x - a|^2 a}{1 - 2ax + |a|^2|x|^2} \tag{4.7}$$

where $ax = a_1x_1 + a_2x_2 + \cdots + a_nx_n$. Then $\gamma_a \in \hat{M}(\mathbf{R}^n)$, γ_a maps D onto D, $\gamma_a(a) = 0$ and $\gamma_{-a} \circ \gamma_a = \mathrm{id}_D$ (for proofs see [1]). Furthermore,

$$|\gamma_a(x)| = \frac{|x - a|}{|a||x - a^*|} \tag{4.8}$$

and

$$|\gamma_a'(x)| = \frac{1 - |a|^2}{|a|^2|x - a^*|^2} \tag{4.9}$$

(cf. [1, pp.26, 27, formulae (30) and (32)]).

Now we define, for every $a \in D$, the map $T_a : h \mapsto \tilde{h}$, by

$$\tilde{h}(x) = \frac{|a - a^*|^{n-2}}{|x - a^*|^{n-2}} h(\gamma_a(x)), \quad x \in D \tag{4.10}$$

for $a \neq 0$, and $T_0 = \mathrm{id}$. By using (4.9), this can be rewritten as

$$\tilde{h}(x) = \frac{|a - a^*|^{n-2}}{|a|^{n-2}} (1 - |a|^2)^{\frac{n-2}{2}} |\gamma_a'(x)|^{\frac{n-2}{2}} h(\gamma_a(x)).$$

By Lemma 4.1, we see that T_a preserves harmonicity and positivity. Moreover, if $h(0) = 1$, then $\tilde{h}(a) = 1$. If for $b \in D$, $\mathcal{H}_b = \{h : h \in \mathcal{H},\ h(b) = 1\}$, then the discussion above shows that $T_a : \mathcal{H}_0 \to \mathcal{H}_a$.

Now we show that T_a is onto. Let $u \in \mathcal{H}_a$ and define

$$h(x) = \frac{|a^*|^{n-2}}{|x + a^*|^{n-2}} u(\gamma_{-a}(x)). \tag{4.11}$$

Then h is positive and harmonic in D, and $h(0) = u(\gamma_{-a}(0)) = u(a) = 1$. Further,

$$T_a h(x) = \frac{|a - a^*|^{n-2}}{|x - a^*|^{n-2}} h(\gamma_a(x))$$

$$= \frac{|a - a^*|^{n-2} |a^*|^{n-2}}{|x - a|^{n-2} |\gamma_a(x) + a^*|^{n-2}} u(\gamma_{-a}(\gamma_a(x)))$$

$$= \left(\frac{|a - a^*||a^*|}{|x - a||\gamma_a(x) + a^*|} \right)^{n-2} u(x) = u(x).$$

The last equation follows from an easy computation showing that the factor before $u(x)$ is identically 1.

Now we are ready to compute s_a.

$$s_a(x) = \inf\{u(x) : u \in \mathcal{H}_a\} = \inf\{(T_a h)(x) : h \in \mathcal{H}_0\}$$

$$= \inf\left\{ \frac{|a - a^*|^{n-2}}{|x - a^*|^{n-2}} h(\gamma_a(x)) : h \in \mathcal{H}_0 \right\} = \frac{|a - a^*|^{n-2}}{|x - a^*|^{n-2}} s_0(\gamma_a(x))$$

Now we use (4.4) and (4.8) and get

$$s_a(x) = \frac{|a-a^*|^{n-2}}{|x-a^*|^{n-2}} \frac{1-|\gamma_a(x)|^2}{(1+|\gamma_a(x)|)^n}$$

$$= \frac{|a-a^*|^{n-2}}{|x-a^*|^{n-2}} \frac{1-\frac{|x-a|^2}{|a|^2|x-a^*|^2}}{\left(1+\frac{|x-a|}{|a||x-a^*|}\right)^n} = (1-|a|^2)^{n-2} \frac{|a|^2|x-a^*|^2-|x-a|^2}{(|a||x-a^*|+|x-a|)^n}.$$

Since $|a|^2|x-a^*|^2 - |x-a|^2 = (1-|a|^2)(1-|x|^2)$, we finally obtain

$$s_a(x) = (1-|a|^2)^{n-2} \frac{(1-|a|^2)(1-|x|^2)}{(|a||x-a^*|+|x-a|)^n}, \quad a \neq 0. \tag{4.12}$$

It is obvious from this formula that $\lim_{x\to z} s_a(x) = 0$ for every $z \in \partial D$. Therefore, s_a is a potential. The Riesz measure of s_a is the measure with density $-\Delta s_a$ (with respect to the Lebesgue measure on D). We will spare the reader of several pages of computation and simply write down the formulae:

$$-\Delta s_0(x) = \frac{n(n-1)}{|x|(1+|x|)^2} s_0(x), \quad x \neq 0 \tag{4.13}$$

and, for $a \neq 0$,

$$-\Delta s_a(x) = \frac{n(n-1)(1-|a|^2)^n}{|a||x-a^*||x-a|(|a||x-a^*|+|x-a|)^2} s_a(x), \quad x \neq a. \tag{4.14}$$

According to Proposition 2.4, for any $x \in D$ there exists a point $z = z(0,x)$ on the boundary ∂D such that $s_0(x) = P(x,z)$. It is obvious from the way how s_0 was computed that $z = -x/|x|$ for $x \neq 0$, i.e., z is the point on the diameter through x.

If we are given $z \in \partial D$, then the set of points x in D for which $s_0(x) = P(x,z)$ is the segment $[0,-z) = \{\lambda z : -1 < \lambda \leq 0\}$. Let us denote this segment by $L_{0,z}$. Thus, $L_{0,z} = \{x \in D : s_0(x) = P(x,z)\}$.

There is nothing special about the point 0, so we fix a point $a \in D$, $a \neq 0$. Again by Proposition 2.4, for any $x \in D$ there is a point $\zeta = \zeta(a,x) \in \partial D$ such that $s_a(x) = P(x,\zeta)/P(a,\zeta)$. To find such ζ we go backwards . We fix $\zeta \in \partial D$ and determine the set $L_{a,\zeta} = \{x \in D : s_a(x) = P(x,\zeta)/P(a,\zeta)\}$.

Let γ_a be as in (4.7). We obtained that

$$s_a(x) = \frac{|a-a^*|^{n-2}}{|x-a^*|^{n-2}} s_0(\gamma_a(x))$$

for every $x \in D$. Let $z = \gamma_a(\zeta) \in \partial D$ and choose x such that $\gamma_a(x) \in L_{0,z} = (0, -z)$. Then $s_0(\gamma_a(x)) = P(\gamma_a(x), z)$, and so

$$s_a(x) = \frac{|a-a^*|^{n-2}}{|x-a^*|^{n-2}} P(\gamma_a(x), \gamma_a(\zeta)) = \frac{|a-a^*|^{n-2}}{|x-a^*|^{n-2}} \frac{1-|\gamma_a(x)|^2}{|\gamma_a(x)-\gamma_a(\zeta)|^n}.$$

By formula (15), p.19, in [1], $|\gamma_a(x) - \gamma_a(\zeta)| = |\gamma_a'(x)|^{1/2} |\gamma_a'(\zeta)|^{1/2} |x - \zeta|$. A simple computation using the above, (4.8), and (4.9) determines $s_a(x) = P(x,\zeta)/P(a,\zeta)$.

This shows that $L_{a,\zeta} = \gamma_a^{-1}(L_{0,z}) = \gamma_{-a}(L_{0,z})$, where $z = \gamma_a(\zeta)$. Conformal property of γ_a easily implies that $L_{a,\zeta}$ is an arc of the circle through a, perpendicular to the sphere ∂D. Precisely, if ζ is given, let S be the circle through a perpendicular to the sphere ∂D. Then $S \cap \partial D = \{\zeta, \zeta_1\}$. The arc of S connecting a and ζ_1 is $L_{a,\zeta}$.

Therefore, if $x \in D$, let S be the circle through a and x perpendicular to ∂D. The point $\zeta \in \partial D$ such that $s_a(x) = P(x,\zeta)/P(a,\zeta)$ is the point on $S \cap \partial D$ such that x does not belong to the arc connecting a and ζ. Note that ζ is unique.

Let us denote $s_a(x)$ by $k(x,a)$ as in Chapter 2. For $n = 2$,

$$k(x,a) = \frac{(1-|a|^2)(1-|x|^2)}{(|a||x-a^*| + |x-a|)^2}$$

which is symmetric in x and a due to the fact that $|a||x-a^*| = |x||a-x^*|$. When $n \geq 3$, k is *not* symmetric.

One can consider quotients of the form $s_a(x)/s_0(x)$. Then straightforward computation shows that, for every $z \in \partial D$,

$$\lim_{x \to z} \frac{s_a(x)}{s_0(x)} = P(x,z),$$

which was the motivation for Section 2.2.

It is tempting to compare the function $k(x,a)$ with the familiar Green function $G(x,a)$ for the unit ball D. Recall the formulae for the Green function:

$$G(x,a) = \log\left|\frac{1-x\bar{a}}{x-a}\right|, \quad n=2 \tag{4.15}$$

$$G(x,a) = \frac{1}{|x-a|^{n-2}} - \frac{1}{|x|^{n-2}}\frac{1}{|x-a^*|^{n-2}}, \quad n \geq 3 \tag{4.16}$$

where $\bar{a}$ is the conjugate of a, and $a^* = Ja$ (e.g. [23, 6.37]).

Green function is infinite on the diagonal while k is 1 on the diagonal. Therefore, it doesn't seem that G and k can be compared away from the boundary ∂D. So, let us consider both function when one of the variables is near the boundary.

First we let $n = 2$. Then

$$\frac{k(0,a)}{G(0,a)} = \frac{1-|a|}{1+|a|}\frac{1}{\log|1/a|} = \frac{1}{1+|a|}\frac{\log|a|}{|a|-1}.$$

Since $(|a|-1)/\log|a| \to 1$ as $a \to z \in \partial D$, it follows

$$\lim_{a\to z}\frac{k(0,a)}{G(0,a)} = \frac{1}{2}, \quad z \in \partial D. \tag{4.17}$$

For $x \in D$, both $G(x,a)/G(0,a)$ and $k(x,a)/k(0,a)$ converge to $P(x,z)$ as $a \to z$. Hence, from (4.16) it follows

$$\lim_{a\to z}\frac{k(x,a)}{G(x,a)} = \frac{1}{2}, \quad z \in \partial D. \tag{4.18}$$

For $n \geq 3$ the situation is different. First we consider $k(x,a)/G(x,a)$ as $x \to z \in \partial D$. Straightforward computation using (4.12) and (4.16) gives

$$\lim_{x\to z}\frac{k(x,a)}{G(x,a)} = \frac{(1-|a|^2)^{n-2}}{2^{n-1}(n-2)}, \quad z \in \partial D. \tag{4.19}$$

This shows that when x is close to the boundary, the infimum $s_a(x)$ is comparable with the Green function $G(x,a)$. On the other hand, since

$$\lim_{a\to z}\frac{k(x,a)}{k(a,x)} = \lim_{a\to z}\frac{(1-|a|^2)^{n-2}}{(1-|x|^2)^{n-2}} = 0,$$

it follows that

$$\lim_{a\to z}\frac{k(x,a)}{G(x,a)}=\lim_{a\to z}\frac{k(x,a)}{k(a,x)}\frac{k(a,x)}{G(a,x)}=0,\quad z\in\partial D. \tag{4.20}$$

Therefore, for a fixed $x\in D$, when a approaches the boundary, $s_a(x)$ goes to zero faster then $G(x,a)$. This suggests that the fact that $k(x,a)/k(x_0,a)$ tends to the kernel function $K(x,z)$ as $a\to z\in\partial D$ (here D is Lipschitz, and $x_0\in D$), is not obtainable from the corresponding result for the Green function.

We also note that $k(x,a)\sim\delta(a)^{n-1}$ when a is near the boundary ∂D, and $k(x,a)\sim\delta(x)$ when x is near ∂D, where $\delta(\cdot)$ denotes the distance to the boundary.

4.2 Function k for Some Other Domains

In this section we use transformations that preserve harmonicity to obtain explicit formulae and some properties of s_a for certain domains.

Let $B(x_0,r)$ be the ball in $\mathbf{R}^n$ centered at x_0 with radius r. The *reflection through the sphere* $S(x_0,r)=\{x\in\mathbf{R}^n:|x-x_0|=r\}$ is the mapping J defined by

$$J(x)=x^*=x_0+\frac{r^2}{|x-x_0|^2}\,(x-x_0). \tag{4.21}$$

Obviously $J^2=\mathrm{id}_{\mathbf{R}^n}$.

Let D be an open set in $\mathbf{R}^n\setminus\{x_0\}$ and let $D^*=J(D)$. The *Kelvin transform* of a function h on D is a function h^* on D^* defined by

$$h^*(x^*)=\frac{r^{n-2}}{|x^*-x_0|^{n-2}}\,h(x). \tag{4.22}$$

Kelvin transformation preserves positivity and harmonicity (e.g., [23, 4.25]). Let $\mathcal{H}(D)$ and $\mathcal{H}(D^*)$ denote families of positive harmonic functions on D and D^* respectively. For $a\in D$ we define the mapping $S_a:\mathcal{H}(D)\to\mathcal{H}(D^*)$ by

$$(S_ah)(x^*)=\frac{|a^*-x_0|^{n-2}}{|x^*-x_0|^{n-2}}\,h(x). \tag{4.23}$$

Then $S_a h \in \mathcal{H}(D^*)$ and $(S_a h)(a^*) = 1$. It is easy to see that S_a is $1-1$ and onto. Therefore, as in the previous section, we get the formula

$$s_{a^*}(x^*) = \frac{|a^* - x_0|^{n-2}}{|x^* - x_0|^{n-2}}\, s_a(x). \tag{4.24}$$

Further, in the same way as in Section 4.1, one can compute the function s_a for the ball $B(a,\rho)$. It follows

$$s_a(x) = \rho^{n-2} \frac{\rho^2 - |x-a|^2}{(\rho + |x-a|)^n}. \tag{4.25}$$

Now we compute the function s_a for the upper halfspace. We work in the $(n+1)$-dimensional Euclidean space $\mathbf{R}^{n+1}$. Points in $\mathbf{R}^{n+1}$ are denoted by $P, Q, R, \ldots$, $P = (x,y)$, $x \in \mathbf{R}^n$, $y \in \mathbf{R}$. Let $H^+ = \{(x,y) \in \mathbf{R}^{n+1} : y > 0\}$ be the upper halfspace. We fix the point $Q^* \in H^+$, $Q^* = (0,v)$, $v > 0$. The goal is to compute s_{Q^*}.

Let J denote the reflection through the sphere $S(-Q^*, v)$, and let D be the ball $B(-Q^*/2, v/2)$. Then $D^* = H^+$ and an easy computation gives $J(-Q^*/2) = Q^*$. Let $P^* \in H^+$; then, by (4.24) and (4.25),

$$\begin{aligned} s_{Q^*}(P^*) &= \frac{|Q^* - (-Q^*)|^{n-1}}{|P - (-Q^*)|^{n-1}}\, s_{-Q^*/2}(P) \\ &= \frac{2|Q^*|}{|P + Q^*|^{n-1}} \left(\frac{v}{2}\right)^{n-1} \frac{|-Q^*/2|^2 - |P + Q^*/2|^2}{(|-Q^*/2| + |P + Q^*/2|)^{n+1}}. \end{aligned}$$

After an easy computation one gets

$$s_{Q^*}(P) = \frac{v^{2n-2}}{|P^* + Q^*|^{n-1}} \frac{v^2/4 - |P + Q^*/2|^2}{(v/2 + |P + Q^*/2|)^{n+1}}. \tag{4.26}$$

Let $P = (x,y)$, $x \in \mathbf{R}^n$, $y > 0$. Straightforward computations yield the following formulae:

$$|P^* + Q^*| = [|x|^2 + (y+v)^2]^{1/2} \tag{4.27}$$

$$P = J(P^*) = \frac{1}{|x|^2 + (y+v)^2} \left(v^2 x, -\frac{v}{2}(|x|^2 + (y+v)^2) + v^2(y+v) \right) \tag{4.28}$$

$$|P + Q^*/2|^2 = \frac{v^2}{4} \frac{|x|^2 + (y-v)^2}{|x|^2 + (y+v)^2} \tag{4.29}$$

$$\frac{v^2}{4} - |P + Q^*/2|^2 = \frac{v^3 y}{|x|^2 + (y+v)^2}. \tag{4.30}$$

By putting together (4.27)-(4.30) - and after some simplification - we finally obtain

$$s_{Q^*}(P^*) = \frac{2^{n+1}\, v^n y}{(\sqrt{|x|^2 + (y+v)^2} + \sqrt{|x|^2 + (y-v)^2})^{n+1}} \tag{4.31}$$

where $(P^*) = (x, y)$ and $Q^* = (0, v)$.

If $Q^* = (u, v)$, $u \in \mathbf{R}^n$, then one should replace x by $x - u$ in (4.31) to obtain the formula for s_{Q^*}. When we are in the two-dimensional space, that formula gives the symmetry in P^* and Q^*.

One can again consider the limit of the quotient $s_{Q^*}(P^*)/s_{Q^*}({P_0}^*)$ as Q^* tends to some point $(t, 0)$ on the boundary of H^+. Here ${P_0}^* = (x_0, y_0)$ is some fixed point in H^+. The limit is equal to

$$\frac{y}{y_0} \frac{(|x_0 - t|^2 + {y_0}^2)^{(n+1)/2}}{(|x - t|^2 + y^2)^{(n+1)/2}},$$

which is a constant multiple of the kernel function

$$\frac{y}{(|x - t|^2 + y^2)^{(n+1)/2}}, \quad x, t \in \mathbf{R}^n,\ y > 0,$$

for the upper halfspace H^+. Finally we note that as Q^* goes to infinity (i.e., $v \to \infty$), the quotient $s_{Q^*}(P^*)/s_{Q^*}({P_0}^*)$ approaches y/y_0.

Now we consider domains in $\mathbf{R}^2$. We identify $\mathbf{R}^2$ with the complex plane $\mathbf{C}$.

Let $D = \{z \in \mathbf{C} : |z| < 1\}$ be the unit disc in $\mathbf{C}$ and let D_1 be a simply connected region in $\mathbf{C}$ (with at least two boundary points). Then there is a conformal mapping $w : D_1 \to D$. Let $w = u + iv$. For $f : D \to \mathbf{R}$ of class C^2, we define $g : D_1 \to \mathbf{R}$ by

$$g(x) = f(w(x)), \quad x \in D_1. \tag{4.32}$$

Then $g \in \mathcal{C}^2(D_1)$ and

$$\Delta g(x) = |\nabla u(x)|^2 (\Delta f)(w(x)). \tag{4.33}$$

(e.g. [23, 6.19]).

If f is positive and harmonic in D, i.e., $f \in \mathcal{H}(D)$, then g is positive and harmonic in D_1, $g \in \mathcal{H}(D_1)$. This follows from definition and (4.33). Moreover, $g(x) = 1$ if and only if $f(w(x)) = 1$.

For $a \in D_1$, let $\tilde{s}_a(y) = \inf\{g(y) : g \in \mathcal{H}_a(D_1)\}$. Then

$$\tilde{s}_a(x) = \inf\{f(w(x)) : f \in \mathcal{H}_{w(a)}(D)\} = s_{w(a)}(w(x)). \tag{4.34}$$

Therefore, once w is known, one can obtain the formula for $\tilde{s}_a$.

Let $\tilde{k}(x,a) = \tilde{s}_a(x)$. Then, from (4.34), we see that $\tilde{k}$ is symmetric in x and a.

4.3 Examples

It is not easy to find nontrivial examples of functions in $\mathcal{H}^{\text{inf}}$ on a given domain D. By trivial we mean functions that are explicitly written as an infimum of positive harmonic functions. The other class of functions in $\mathcal{H}^{\text{inf}}$ which may be regarded as trivial are positive concave functions. Yet, they are useful, because we can use them to get some nontrivial examples.

Note, on the other hand, that we do have a necessary condition for a potential to be in $\mathcal{H}^{\text{inf}}$. Let G_D denote the Green function on D, and, for a positive measure μ on D, let

$$G_D\mu(x) = \int_D G_D(x,y)\mu(dy).$$

If $G_D\mu \in \mathcal{H}^{\text{inf}}$, then μ cannot have compact support (Proposition 3.3). Of course, this condition is far from being sufficient. There are measures without compact support whose potentials are not even continuous. It seems that the characterization of measures which give potentials in $\mathcal{H}^{\text{inf}}$ is difficult, if not impossible.

In this section we give some examples of functions in $\mathcal{H}^{\inf}$. We start with an auxiliary result.

Let $D_1 \subset \mathbf{R}^{n_1}$, $D_2 \subset \mathbf{R}^{n_2}, \ldots,$ $D_k \subset \mathbf{R}^{n_k}$ be bounded domains and let $D = D_1 \times D_2 \times \ldots \times D_k$ be a bounded domain in $\mathbf{R}^{n_1+n_2+\ldots n_k}$. Assume that $u_i \in \mathcal{H}^{\inf}(D_i)$, $i = 1, 2, \ldots, k$. Then the function u defined by

$$u(x) = u(x_1, x_2, \ldots, x_k) = u_1(x_1)u_2(x_2)\ldots u_k(x_k)$$

is in $\mathcal{H}^{\inf}(D)$. To see this, we fix x in D. For each i, there is a harmonic function h_i on D_i such that $h_i \geq u_i$ and $h_i(x_i) = u_i(x_i)$ (see Lemma 2.7). We define h on D by

$$h(y) = h(y_1, y_2, \ldots, y_k) = h_1(y_1)h_2(y_2)\ldots h_k(y_k).$$

Then h is harmonic on D, $h \geq u$, and $h(x) = u(x)$. Since such an h can be found for any x in D, it follows that $u \in \mathcal{H}^{\inf}(D)$.

The above can be used to get some examples on product domains in $\mathbf{R}^n$. We assume that $D = (a_1, b_1) \times \cdots \times (a_n, b_n)$. Let

$$\phi_i(x_i) = \sin\left(\pi \frac{x_i - a_i}{b_i - a_i}\right), \quad i = 1, \ldots, n.$$

Then ϕ_i is the eigenfunction for d^2/dx_i^2 corresponding to the eigenvalue $-\pi^2/(b_i - a_i)^2$. Since ϕ_i is concave, it is in $\mathcal{H}^{\inf}(a_i, b_i)$. Let ϕ be defined on D by

$$\phi(x) = \phi(x_1, \ldots, x_n) = \phi_1(x_1)\ldots\phi_n(x_n).$$

Then ϕ is the eigenfunction of Δ corresponding to the eigenvalue $\sum_{i=1}^n(-\pi^2/(b_i - a_i)^2)$. Since ϕ is positive, it is the first eigenfunction of Δ on D. By the result above, ϕ is in $\mathcal{H}^{\inf}(D)$. Note that ϕ is not a concave function.

Let (X_t, P^x) be the Brownian motion on $\mathbf{R}^n$ and let τ_D denote the first exit time from D,

$$\tau_D = \inf\{t > 0 : X_t \notin D\}.$$

Let $\psi(x) = P^x(\tau_D > t)$. It is well-known that ψ is excessive. We show that ψ is in $\mathcal{H}^{\inf}(D)$.

Let $\tau_i = \inf\{t > 0 : X_t^{(i)} \notin (a_i, b_i)\}$, $i = 1, \ldots, n$, where $X_t^{(i)}$ denotes the i-th component of X. Since $X^{(i)}$ are independent processes, times τ_i are independent. Further, $\tau_D = \tau_1 \wedge \ldots \wedge \tau_n$. Hence,

$$\psi(x) = P^x(\tau_D > t) = P^x(\tau_1 > t) \ldots P^x(\tau_n > t) = P^{x_1}(\tau_1 > t) \ldots P^{x_n}(\tau_n > t),$$

where $x = (x_1, \ldots, x_n)$. But each $x_i \mapsto P^{x_i}(\tau_i > t)$ is excessive for the 1-dimensional Brownian motion $X^{(i)}$, and is therefore an element of $\mathcal{H}^{\inf}(a_i, b_i)$. Since ψ is a product of such functions, it follows that ψ is in $\mathcal{H}^{\inf}(D)$.

Let D be any bounded domain in $\mathbf{R}^n$ and let τ_D be the exit time from D of the n-dimensional Brownian motion. We show that the function $\phi(x) = E^x(\tau_D)$, i.e., the expected exit time from D is in $\mathcal{H}^{\inf}(D)$.

First we establish this result for the ball $B = B(0, r)$. Let $\tau_B = \inf\{t > 0 : X_t \notin B\}$. Then

$$E^x(\tau_B) = \frac{1}{n}(r^2 - |x|^2), \quad x \in B$$

(e.g. [23, 4.6]). The function $x \mapsto r^2 - |x|^2$ is concave on B, and therefore in $\mathcal{H}^{\inf}(B)$.

Now we take up the general case. Since D is bounded, there is a ball $B = B(0, r)$ such that $D \subset B$. Let $\phi(x) = E^x(\tau_D)$, $x \in D$, and $\psi(x) = E^x(\tau_B)$, $x \in B$. It is known that $\Delta\phi = -1$ in D, and $\Delta\psi = -1$ in B. Hence $\Delta(\psi - \phi) = 0$ in D, so $\psi - \phi$ is harmonic in D. Obviously, $\phi \leq \psi$ in D. If $h = \psi - \phi$, then h is strictly positive and harmonic. Since ψ is in $\mathcal{H}^{\inf}(B)$, certainly $\psi \in \mathcal{H}^{\inf}(D)$. Hence, $\psi = \inf_\alpha h_\alpha$, h_α harmonic and positive in D. But then $\phi = \psi - h = \inf_\alpha(h_\alpha - h)$ and each $h_\alpha - h$ is positive and harmonic in D.

Let (P_t) be the semigroup of the Brownian motion in $\mathbf{R}^n$. Then, for nonnegative measurable f,

$$P_t f(x) = \int_{\mathbf{R}^n} p(t,x,y) f(y) dy,$$

where

$$p(t,x,y) = \frac{1}{(2\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{2t}\right).$$

It is easy to see that the function $y \mapsto \exp(-|y|^2/2t)$ is concave in the ball $B(0,\sqrt{t})$. Therefore, $y \mapsto p(t,x,y)$ is concave in $B(x,\sqrt{t})$ and $x \mapsto p(t,x,y)$ is concave in $B(y,\sqrt{t})$.

Let us fix $T > 0$ and let $t \geq 4T^2$. If $y \in B(0,T)$, then $|x-y| < 2T$ for each $x \in B(0,T)$. Therefore $x \in B(y,2T) \subset B(y,\sqrt{t})$. This proves that $x \mapsto p(t,x,y)$ is concave in $B(0,T)$ for each $y \in B(0,T)$. In particular, $x \mapsto p(t,x,y)$ is an element of $\mathcal{H}^{\text{inf}}$ on $B(0,T)$ for any $y \in B(0,T)$.

Let μ be a positive Radon measure on $B(0,t)$. By using a result from 2.1 we see that the function $x \mapsto \int_{B(0,T)} p(t,x,y)\mu(dy)$ is in $\mathcal{H}^{\text{inf}}$ on $B(0,T)$. In particular, if f is nonnegative and $f \in \mathcal{L}^1_{\text{loc}}(B(0,T))$, then $x \mapsto \int_{B(0,T)} p(t,x,y)f(y)(dy) \in \mathcal{H}^{\text{inf}}(B(0,T))$.

Let D be a bounded domain in $\mathbf{R}^n$ and let f be a nonnegative function which is identically zero outside D and $f \in \mathcal{L}^1_{\text{loc}}(D)$. Let T be large enough so that $D \subset B(0,T)$ and let $t \geq 4T^2$. Then, for $x \in D$,

$$P_t f(x) = \int_{\mathbf{R}^n} p(t,x,y)f(y)dy = \int_{B(0,T)} p(t,x,y)f(y)dy,$$

and from above it follows that $P_t f \in \mathcal{H}^{\text{inf}}(D)$.

Further, $t \mapsto P_t f(x)$ is continuous, so

$$x \mapsto \int_{4T^2}^{\infty} P_t f(x)\, dt$$

is in $\mathcal{H}^{\text{inf}}(D)$ (provided that it is not identically $+\infty$). Let

$$Gf(x) = \int_0^\infty P_t f(x)\, dt.$$

Then

$$\int_{4T^2}^\infty P_t f(x) = P_{4T^2} G f(x) = G P_{4T^2} f(x).$$

Hence $GP_{4T^2} f \in \mathcal{H}^{\text{inf}}(D)$, and similarly, for each $t > 4T^2$, $GP_t f(x) \in \mathcal{H}^{\text{inf}}(D)$.

Let G_D denote the Green function for D. If u is nonnegative function, then it is well known that $G_D u = Gu - h$, where h is a positive harmonic function on D (e.g. [23, 6.2]). By using this fact and the above result, it follows that for nonnegative $f \in \mathcal{L}^1_{\text{loc}}(D)$ there exists $T > 0$ such that for every $t \geq 4T^2$, $G_D(P_t f) \in \mathcal{H}^{\text{inf}}(D)$.

4.4 Two Counterexamples

In this section we give two examples. One of them shows that $\mathcal{H}^{\text{inf}}$ is not a potential cone, and the other that $\mathcal{H}^{\text{inf}}$ is not a function cone. In both cases we take for our domain the unit disc in $\mathbf{R}^2$.

To say that $\mathcal{H}^{\text{inf}}$ is not a potential cone is rather vague, since we have not defined what is meant by this. There are different approaches and axiomatizations of potential theory. We shall adopt the one from [2] and show that $\mathcal{H}^{\text{inf}}$ is not a balayage space. Following [2, pp. 56-57], we consider a domain D and a convex cone $\mathcal{W}$ of positive lower semicontinuous functions on D. The ($\mathcal{W}-$) *fine topology* on D is the coarsest topology on D which is finer than the initial topology and for which all functions in $\mathcal{W}$ are continuous. Then $(D, \mathcal{W})$ is said to be a *balayage space* if the following axioms are satisfied:

(B_1) If $\{u_n\}$ is an increasing sequence of functions in $\mathcal{W}$, then $\lim_n u_n \in \mathcal{W}$.

(B_2) $\widehat{\inf_{\alpha \in \mathcal{A}} u_\alpha}^f \in \mathcal{W}$ for any subfamily $\{u_\alpha : \alpha \in \mathcal{A}\}$, where $\widehat{\ }^f$ denotes the fine l.s.c. regularization.

(B_3) If $u, v_1, v_2 \in \mathcal{W}$ such that $u \leq v_1 + v_2$ there exist $u_1, u_2 \in \mathcal{W}$ such that $u = u_1 + u_2$, $u_1 \leq v_1$, $u_2 \leq v_2$.

(B_4) There exists a function cone $\mathcal{P} \subset \mathcal{C}^+(D)$ such that every function $u \in \mathcal{W}$ is an increasing limit of functions in $\mathcal{P}$.

For u in $\mathcal{W}$ and any subset A of D let

$$R_u^A = \inf\{v \in \mathcal{W} : v \geq u \text{ on } A\}.$$

It is proved in [2, p.243, Prop.1.1], that if $A \subset D$ and $u, v \in \mathcal{W}$, then

$$R_{u+v}^A = R_u^A + R_v^A. \tag{4.35}$$

We take for D the unit disc in $\mathbf{R}^2$ and consider the convex cone $\mathcal{H}^{\text{inf}}$ on D. Then $(D, \mathcal{H}^{\text{inf}})$ satisfies axioms (B_1),(B_2) and (B_4) (for (B_4) see Lemma 2.9). Since (B_3) is difficult to disprove directly, we show that (4.35) does not hold.

Let $a = (-1/2, 0)$, $b = (1/2, 0)$ be two points in D and let $A = \{a, b\}$. Then $R_{s_a}^A = \inf\{u \in \mathcal{H}^{\text{inf}} : u(a) \geq s_a(a), u(b) \geq s_a(b)\} = s_a$, and similarly, $R_{s_b}^A = s_b$. Hence, $R_{s_a}^A + R_{s_b}^A = s_a + s_b$. Further,

$$R_{s_a+s_b}^A = \inf\{u \in \mathcal{H} : u(a) \geq s_a(a) + s_b(a), u(b) \geq s_a(b) + s_b(b)\}$$

$$= \inf\{u \in \mathcal{H} : u(a) \geq \gamma, u(b) \geq \gamma\},$$

where $\gamma = s_a(a) + s_b(a) = s_a(b) + s_b(b)$ because of symmetry. Now we use formula (4.12) to compute s_a and s_b. It follows that $s_a(b) = 1/9$, so $\gamma = 10/9$.

Let $x_0 = (0, 1/2)$; then, by (4.12) and by symmetry,

$$s_a(x_0) = s_b(x_0) = \frac{9}{(\sqrt{17} + \sqrt{8})^2} \approx 0.1862436, \tag{4.36}$$

and so

$$R_{s_a}^A(x_0) + R_{s_b}^A(x_0) = s_a(x_0) + s_b(x_0) = \frac{18}{(\sqrt{17} + \sqrt{8})^2} \approx 0.3724872\ . \tag{4.37}$$

Now we find a function $h \in \mathcal{H}$ such that $h(a) \geq \gamma$, $h(b) \geq \gamma$, but $h(x_0) < s_a(x_0)+s_b(x_0)$. This will show that $R^A_{s_a+s_b}(x_0) < R^A_{s_a}(x_0)+R^A_{s_b}(x_0)$, which contradicts (4.35).

Let $z_1 = (24/25, -7/25)$ and $z_2 = (-24/25, -7/25)$ be two points on the boundary ∂D, and let $P(x, z)$ denote the Poisson kernel. Then

$$P(a, z_1) = P(b, z_2) = 75/221, \quad P(a, z_2) = P(b, z_1) = 75/29.$$

Let $\beta = P(a, z_1) + P(a, z_2) = P(b, z_1) + P(b, z_2)$, and let

$$h(x) = \frac{\gamma}{\beta}(P(x, z_1) + P(x, z_2)).$$

Then $h(a) = h(b) = \gamma$. Further, $P(x_0, z_1) = P(x_0, z_2) = 25/51$, so that

$$h(x_0) = \frac{12818}{34425} \approx 0.3723456 \ . \tag{4.38}$$

By comparing (4.37) and (4.38) we see that $h(x_0)$ is strictly smaller than $R^A_{s_a}(x_0) + R^A_{s_b}(x_0)$. This proves that $(D, \mathcal{H}^{\inf})$ is not a balayage space.

Now we show that $\mathcal{H}^{\inf}$ is not a function cone as defined in Section 2.1, because the axiom (F_3) does not hold.

Assume, on the contrary, that it holds. Then for every $v \in \mathcal{H}^{\inf}$, there is $u \in \mathcal{H}^{\inf}$, such that for every $\epsilon > 0$ there is a compact set K in D with $v(x) \leq \epsilon u(x)$ for every $x \in D \setminus K$.

Let us take $v = P(\cdot, z)$, where $z = (1, 0) \in \partial D$. Then for u as above we take $\epsilon < \frac{1}{2u(0)}$ and choose a compact set K satisfying $P(x, z) \leq \epsilon u(x)$ for every $x \in D \setminus K$. Since u is in $\mathcal{H}^{\inf}$, it satisfies for each $x \in D$ Harnack inequality

$$u(x) \leq \frac{1 - |x|^2}{(1 - |x|)^2} u(0).$$

Then

$$P(x, z) = \frac{1 - |x|^2}{|z - x|^2} \leq \epsilon u(x) \leq \frac{1 - |x|^2}{(1 - |x|)^2} \epsilon u(0) < \frac{1}{2} \frac{1 - |x|^2}{(1 - |x|)^2}, \quad x \in D \setminus K$$

which implies

$$\frac{(1-|x|)^2}{|z-x|^2} < \frac{1}{2}, \quad \text{for every } x \in D \setminus K.$$

We may take $x = (\alpha, 0)$ which is in $D \setminus K$ when α is close to 1. Then the left-hand side above is 1, which yields a contradiction.

4.5 On Extremal Functions

Let D be the unit ball in $\mathbf{R}^n$. Nonnegative superharmonic functions on D form a convex cone. The description of extremal functions in this cone is well known: The only extremal functions are $P(\cdot, z)$, $z \in \partial D$, and $G_D(\cdot, y)$, $y \in D$, where P denotes the Poisson kernel and G_D the Green function for D. If, instead of the cone of superharmonic functions we consider $\mathcal{H}^{\inf}$ on D, functions $P(\cdot, z)$, $z \in \partial D$, are still extremal. On the other hand, $G_D(\cdot, y)$ are not even in $\mathcal{H}^{\inf}$. Hence, they cannot be extremal. We saw in Section 2.1 that $s_y = k(\cdot, y)$, $y \in D$, are extremal functions in $\mathcal{H}^{\inf}$. In this section we give some other examples of extremal functions. We do not know the complete description of extremal elements in $\mathcal{H}^{\inf}$.

For a point $\zeta \in D$ let $-\zeta$ denote the point on ∂D opposite to ζ, $(0, \zeta) = \{\lambda\zeta : 0 < \zeta < 1\}$ and $[0, \zeta) = \{\lambda\zeta : 0 \le \zeta < 1\}$. Recall the inequality (4.1):

$$\frac{1-|x|^2}{(1+|x|)^n} h(0) \le h(x).$$

Let $z \in \partial D$; if $x \in (0, -z)$, then

$$P(x,z) = \frac{1-|x|^2}{|z-x|^n} = \frac{1-|x|^2}{(1+|x|)^n}.$$

Hence the inequality above can be rewritten as

$$h(x) \ge P(x,z), \quad x \in (0,-z), \; h \in \mathcal{H}_0. \tag{4.39}$$

The following lemma shows that unless $h \equiv P(\cdot, z)$, then the inequality above is strict.

<u>LEMMA 4.2</u> *Let $z \in \partial D$. If $h \in \mathcal{H}$, $h(0) = 1$ such that $h \not\equiv P(\cdot, z)$, then*

$$h(x) > P(x,z), \quad \textit{for every } x \in (0,-z). \tag{4.40}$$

PROOF: If $h = P(\cdot, \zeta)$, where $\zeta \neq z$, then for any $x \in (0,-z)$ we have $|x-\zeta| < |x-z|$. The formula for the Poisson kernel implies (4.40). For an arbitrary $h \in \mathcal{H}$, there is a probability measure μ on ∂D such that $h(x) = \int_{\partial D} P(x,\zeta)\mu(d\zeta)$. Using the fact that for some $\epsilon > 0$, μ charges the set $\{\zeta \in \partial D : P(x,\zeta) > P(x,z) + \epsilon\}$ and by what was shown above, it follows that $h(x) > P(x,z)$. □

<u>Remark:</u> With the same assumptions as in the lemma, one can show that
$h(x) < P(x,z)$, for every $x \in (0,z)$.

<u>LEMMA 4.3</u> *Let $\{h_n\}$ be a sequence of positive harmonic functions in D such that $h_n(0) \geq 1$ and $h_n(x) \to P(x,z)$ for some $x \in (0,-z)$. Then there is a subsequence $\{h_{n_k}\}$ such that $h_{n_k} \to P(\cdot,z)$ everywhere.*

PROOF: The sequence $\{h_n\}$ is bounded at x. Therefore, by Lemma 2.5, there is a subsequence $\{h_{n_k}\}$ and a function h (necessarily harmonic) such that $h_{n_k} \to h$ in $\mathcal{C}(D)$. Further, $h(0) \geq 1$ and $h(x) = \lim_k h_{n_k}(x) = P(x,z)$. By (4.39), $h(x)/h(0) \geq P(x,z)$. Therefore, $h(0) = 1$ and $h(x) = P(x,z)$. Now Lemma 4.2 implies that $h = P(\cdot,z)$.□

<u>LEMMA 4.4</u> *Let $u \in \mathcal{H}^{\text{inf}}$, such that $u(0) \geq 1$ and $u(x) = P(x,z)$ for some $x \in (0,-z)$, $z \in \partial D$. Then $u \leq P(\cdot,z)$ everywhere.*

PROOF: Let $u = \inf_\alpha h_\alpha$, $h_\alpha \in \mathcal{H}$. For x as above, there is a sequence $\{h_n\} \subset \{h_\alpha\}$ such that $u(x) = \lim_n h_n(x)$. Hence $h_n(0) \geq 1$ and $\lim_n h_n(x) = P(x,z)$. By Lemma

4.3, there is a subsequence $\{h_{n_k}\}$ such that $\lim_k h_{n_k} = P(\cdot, z)$ everywhere. Then, for any $y \in D$,

$$u(y) = \inf_\alpha h_\alpha(y) \leq \inf_k h_{n_k}(y) \leq \lim_k h_{n_k}(y) = P(y, z).$$

Thus $u \leq P(\cdot, z)$.□

The next proposition shows that any infimum of Poisson kernels is an extremal element in $\mathcal{H}^{\text{inf}}$.

<u>Proposition 4.1</u> *Let $\{z_i : i \in I\}$ be any family of points on the boundary of D and let*

$$u = \inf_{i \in I} P(\cdot, z_i).$$

Then u is an extremal element in $\mathcal{H}^{\text{inf}}$.

Proof: Assume that $u = f + g$, $f, g \in \mathcal{H}^{\text{inf}}$. Both f and g satisfy inequality (4.1), i.e.,

$$\frac{1 - |x|^2}{(1 + |x|)^n} f(0) \leq f(x) \quad \text{and} \quad \frac{1 - |x|^2}{(1 + |x|)^n} g(0) \leq g(x).$$

Let us fix $i \in I$ and let $x \in (0, -z_i)$. Then

$$f(x) + g(x) = u(x) = P(x, z_i) = \frac{1 - |x|^2}{(1 + |x|)^n} \leq f(x) + g(x)$$

since $f(0) + g(0) = 1$. Hence,

$$f(x) = \frac{1 - |x|^2}{(1 + |x|)^n} f(0) \quad \text{and} \quad g(x) = \frac{1 - |x|^2}{(1 + |x|)^n} g(0), \quad x \in (0, -z_i). \tag{4.41}$$

Let $\tilde{f} = f / f(0)$, $\tilde{g} = g / g(0)$. Then $\tilde{f}(0) = 1$ and $\tilde{f}(x) = P(x, z_i)$ for $x \in (0, -z_i)$. Hence, by Lemma 4.4, $\tilde{f} \leq P(\cdot, z_i)$. Since this holds for every $i \in I$, it follows that $\tilde{f} \leq u$, i.e., $f \leq f(0)u$. Similarly, $g \leq g(0)u$. But, $u = f + g \leq f(0)u + g(0)u = u$, which implies that $f = f(0)u$ and $g = g(0)u$.□

Let us now fix a point $z \in \partial D$. We show that the function $u = P(\cdot, z) \wedge h$ is extremal for every nonnegative bounded harmonic function h satisfying $h(0) = 1$.

Consider the following sets: $U = \{x \in D : P(x,z) < h(x)\}$, $V = \{x \in D : P(x,z) > h(x)\}$, and $W = \{x \in D : P(x,z) = h(x)\}$. If there is an $x \in (0,-z)$ (or $x \in (0,z)$) such that $P(x,z) = h(x)$, then by Lemma 4.2 (i.e., Remark following Lemma), $P(\cdot,z) = h$. Since this case is trivial, we assume that such an x does not exist. Hence, $(0,-z) \subset U$ and $(0,z) \subset V$.

Let us assume that $u = f + g$, with $f,g \in \mathcal{H}^{\text{inf}}$. For $x \in (0,-z)$, an argument similar to the one in the proof of Proposition 4.1 shows that

$$f(x) = \frac{1-|x|^2}{(1+|x|)^n} f(0) \quad \text{and} \quad g(x) = \frac{1-|x|^2}{(1+|x|)^n} g(0),$$

i.e., $f/f(0) = P(\cdot,z) = g/g(0)$ on $(0,-z)$. Lemma 4.4 implies that $f/f(0) \leq P(\cdot,z)$ and $g/g(0) \leq P(\cdot,z)$ everywhere in D. Since $f(0) + g(0) = 1$, this gives that $f(x) = f(0)P(x,z)$ and $g(x) = g(0)P(x,z)$ for $x \in U$. So

$$f = f(0)u \quad \text{and} \quad g = g(0)u \quad \text{in } U. \tag{4.42}$$

By continuity, this holds on $\bar{U} \cap D$.

Note that on V we have $f + g = h$. Hence both f and g are harmonic on V. Because of $\partial U \cap D = \partial V \cap V = W$ and (4.42), it follows that

$$f_{|\partial V \cap D} = f(0)h \quad \text{and} \quad g_{|\partial V \cap D} = g(0)h. \tag{4.43}$$

We would like to show that $f = f(0)h$ and $g = g(0)h$ in V. First we show that f and $f(0)h$, considered as functions on V, have same boundary values at every $\zeta \in \partial V \setminus \{z\}$. By (4.43), this is true on $\partial V \cap D$. If $\zeta \in \partial V \cap \partial D$ and $\zeta \neq z$, let $x \to z$ in V. Since $h < P(\cdot,z)$ in V, it follows that $\lim_{x\to z} h(x) = 0$. But $f \leq h$, so $\lim_{x\to z} f(x) = 0$. Therefore, for every $\zeta \in \partial V \setminus \{z\}$, $\lim_{x\to z} f(x) = \lim_{x\to z} f(0)h(x)$, as x stays in V. The following lemma is needed to conclude that $f = f(0)h$ in V. It can be easily derived using [11, p.118, (8.4)].

<u>LEMMA 4.5</u> *Let V be a bounded domain in $\mathbf{R}^n$ and E a subset of ∂V of harmonic measure zero. If u is a bounded harmonic function in V such that $\lim_{x\to\zeta} u(x) = 0$ for every $\zeta \in \partial V \setminus E$, then $u = 0$ in V.*

We assume that dimension $n \geq 2$. Then the point z is of harmonic measure zero. So Lemma 4.5 is applicable and it follows that $f = f(0)h$ in V. Similarly, $g = g(0)h$ in V. Hence,

$$f = f(0)u \quad \text{and} \quad g = g(0)u \quad \text{in } V. \tag{4.44}$$

Relations (4.42) and (4.44) show that $f = f(0)u$ and $g = g(0)u$ in D. So, the following proposition is proved.

<u>PROPOSITION 4.2</u> *Let h be a bounded nonnegative harmonic function in D such that $h(0) = 1$ and let $z \in \partial D$. Then $u = P(\cdot, z) \wedge h$ is an extremal element in $\mathcal{H}^{\text{inf}}$.*

<u>Remark</u>: Note that the boundedness of h is not essential. We can simply require that $\lim_{x\to z} h(x) = 0$. Then $\lim_{x\to\zeta} f(x) = \lim_{x\to\zeta} f(0)h(x)$ for every $\zeta \in \partial V$, and, consequently, $f = f(0)h$ in V. So, in this case u is also extremal. Examples of such harmonic functions h are functions of the type

$$h(x) = \int_{\partial D} P(x,\zeta)\mu(d\zeta),$$

where μ is singular with respect to the surface measure and the support of μ is bounded away from z.

Yet, some restrictions on h are needed, as the following example shows. Suppose that h_1 is nonnegative harmonic in D with $h_1(0) = 1$. Let us define h by $h = 1/2(P(\cdot, z) + h_1)$. Then $u = P(\cdot, z) \wedge h = \frac{1}{2}P(\cdot, z) + \frac{1}{2}(P(\cdot, z) \wedge h_1)$, hence not extremal.

Up to now, the origin 0 was fixed. As mentioned before, there is nothing special about 0, so we fix some other point $a \in D$, $a \neq 0$. Recall that for $z \in \partial D$, $L_{a,z} = \{x \in D : s_a(x) = P(x,z)/P(a,z)\}$. So, if we replace $(0,-z)$ by $L_{a,z}$, and $P(\cdot,z)$ by $P(\cdot,z)/P(a,z)$, all results of this section are still valid.

The result corresponding to Proposition 4.1 is the following: Let $\{z_i : i \in I\}$ be any family of points on the boundary ∂D, and let

$$u = \inf_{i \in I} \frac{P(\cdot, z_i)}{P(a, z_i)}.$$

Then u is extremal in $\mathcal{H}^{\text{inf}}$.

Let us fix $z \in \partial D$, and let h be a nonnegative bounded harmonic function on D such that $h(a) = 1$. Then the result that corresponds to Proposition 4.2 says that $u = (P(\cdot,z)/P(a,z)) \wedge h$ is extremal. With this fact we can dispose of the condition that $h(0) = 1$ in Proposition 4.2. Indeed, let h be any nonnegative bounded harmonic function on D and let $u = P(\cdot,z) \wedge h$. To avoid trivialities, we assume that there exists $a \in D$ such that $P(a,z) = h(a)$. The function $h/P(a,z)$ is 1 at a, hence $(P(\cdot,z)/P(a,z)) \wedge (h/P(a,z))$ is extremal in $\mathcal{H}^{\text{inf}}$. This implies that u is also extremal. Thus we can state the following result.

<u>Proposition 4.3</u> *Let z be a point on the boundary ∂D and let h be a nonnegative harmonic function on D which is bounded (or such that $\lim_{x \to z} h(x) = 0$). Then the function $u = P(\cdot,z) \wedge h$ is extremal in $\mathcal{H}^{\text{inf}}$.*

CHAPTER 5
HARMONIC FUNCTIONS AND A TIME CHANGE

5.1 On Density of Kernel Functions

In this chapter we depart slightly from the general course and investigate diffusions on a bounded domain D in $\mathbf{R}^n$ which have the same class of harmonic functions as the Brownian motion in D.

We start with a simple but important observation about the Poisson kernel.

Let $x_0 \in \mathbf{R}^n$ and let $0 < r < R$. We denote the Poisson kernel for the ball $B(x_0, R)$ by P:

$$P(x,\zeta) = R^{n-2}\, \frac{R^2 - |x - x_0|^2}{|x-\zeta|^n} \tag{5.1}$$

where $x \in B(x_0, R)$, $\zeta \in S(x_0, R) = \{z \in \mathbf{R}^n : |z - x_0| = R\}$. Let μ be a signed measure on the sphere $S(x_0, R)$ such that

$$\int_{S(x_0,R)} P(x,\zeta)\mu(d\zeta) = 0 \tag{5.2}$$

for every $x \in S(x_0, r)$. The function $h(x) = \int_{S(x_0,R)} P(x,\zeta)\mu(d\zeta)$ is harmonic in $B(x_0, R)$ and by (5.2) it is zero on $S(x_0, r)$. The maximum and the minimum principles imply that $h = 0$ in $B(x_0, r)$ and, therefore, $h = 0$ everywhere in $B(x_0, R)$. Hence, the measure $\mu = 0$.

Let J be the inversion through the sphere $S(x_0, \rho)$ where $\rho = \sqrt{rR}$, i.e.,

$$J(x) = x_0 + \frac{\rho^2}{|x - x_0|^2}(x - x_0). \tag{5.3}$$

The radius ρ is chosen so that J maps $S(x_0, r)$ onto $S(x_0, R)$ and conversely. If $x, y \in S(x_0, r)$, then an easy computation shows that $|x - J(y)| = |J(x) - y|$. Therefore,

$$P(x, J(y)) = P(y, J(x)). \tag{5.4}$$

Let ν be a signed measure on $S(x_0, r)$ and let $\mu = \nu \circ J^{-1}$ be its image measure on $S(x_0, R)$. Assume that for each $\zeta \in S(x_0, R)$

$$\int_{S(x_0,r)} P(x, \zeta)\nu(dx) = 0. \tag{5.5}$$

Then, for each $x \in S(x_0, r)$,

$$\int_{S(x_0,R)} P(x, \zeta)\mu(d\zeta) \;=\; \int_{S(x_0,r)} P(x, J(y))\nu(dy) \;=\; \int_{S(x_0,r)} P(y, J(x))\nu(dy) = 0$$

where the second equality follows from (5.4), and the last from the assumption (5.5) (since $J(x) \in S(x_0, R)$). This shows that μ satisfies (5.2) and therefore $\mu = 0$. So, $\nu = 0$, too. This proves the following.

<u>Lemma 5.1</u> *The linear span of the family* $\{P(\cdot, \zeta)_{|S(x_0,r)} : \zeta \in S(x_0, R)\}$ *is dense in* $\mathcal{C}(S(x_0, r))$.

<u>Proposition 5.1</u> *Let D be a bounded domain in $\mathbf{R}^n$ and let $\mathcal{H}(D)$ denote the family of positive harmonic functions in D. If $B(x_0, r)$ is a ball contained in D, then the linear span of the family $\{h_{|S(x_0,r)} : h \in \mathcal{H}(D)\}$ is dense in $\mathcal{C}(S(x_0, r))$.*

Proof: Let $R > 0$ such that $D \subset B(x_0, R)$ and let P be the Poisson kernel for $B(x_0, R)$. Each function $x \mapsto P(x, \zeta)$ is in $\mathcal{H}(D)$. By Lemma 5.1 the linear span of restrictions of such functions to $S(x_0, r)$ is dense in $\mathcal{C}(S(x_0, r))$. Hence, the linear span of restrictions of functions in $\mathcal{H}(D)$ is also dense. □

Let $\partial_M D$ be the Martin boundary of D, and $\partial_M^\circ D$ the set of minimal points on $\partial_M D$.

<u>COROLLARY 5.1</u> *Let $K(x,\zeta)$ denote the Martin kernel for D, where $x \in D$ and $\zeta \in \partial_M D$. Then the linear span of the family $\{K(\cdot,\zeta)_{|S(x_0,r)} : \zeta \in \partial_M^\circ D\}$ is dense in $\mathcal{C}(S(x_0,r))$.*

PROOF: Let ν be a signed measure on $S(x_0,r)$ such that for every $\zeta \in \partial_M^\circ D$,

$$\int_{S(x_0,r)} K(x,\zeta)\nu(dx) = 0.$$

If h is positive and harmonic, then there exists a unique measure μ on the minimal boundary $\partial_M^\circ D$ such that

$$h(y) = \int_{\partial_M^\circ D} K(y,\zeta)\mu(d\zeta).$$

Hence, by an easy application of the Fubini theorem, it follows that

$$\int_{S(x_0,r)} h(x)\nu(dx) = 0.$$

Proposition 5.2 implies that $\nu = 0$. □

5.2 Application to a Time Change

As in the previous section, D denotes a bounded domain in $\mathbf{R}^n$. Let $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x)$ be the Brownian motion in D killed while exiting D. After being killed, the process goes to "cemetery" Δ. So, we consider the process on the extended state space $D_\Delta = D \cup \{\Delta\}$. The lifetime of X is denoted by ζ.

For a Borel subset B of D_Δ, let T_B denote the hitting time of B,

$$T_B = \inf\{t > 0 : X_t \in B\},$$

and P_B the hitting operator,

$$P_B f(x) = E^x[f(X_{T_B}) : T_B < \infty].$$

Let $A = (A_t)$ be a continuous additive functional of X which is strictly increasing and finite on $[0, \zeta)$, and let τ_t be the inverse of A:

$$\tau_t = \inf\{s : A_s > t\}. \tag{5.6}$$

Let $\hat{X} = (\Omega, \mathcal{F}, \mathcal{F}_{\tau(t)}, X_{\tau(t)}, \theta_{\tau(t)}, P^x)$. It is shown in [3, V(2.11)] that $\hat{X}$ is a standard process.

Let $\tilde{X} = (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{F}}_t, \tilde{X}_t, \tilde{\theta}_t, \tilde{P}^x)$ be a standard Markov process on the state space D_Δ with the lifetime $\tilde{\zeta}$. We assume that paths of $\tilde{X}$ are continuous on $[0, \tilde{\zeta})$. The hitting time to $B \subset D_\Delta$ and the hitting operator are denoted by $\tilde{T}_B$ and $\tilde{P}_B$. To avoid notational complications, we write, for example, $\tilde{E}^x[f(X_{T_B})]$ instead of $\tilde{E}^x[f(\tilde{X}_{\tilde{T}_B})]$.

<u>Proposition 5.2</u> *Assume that for every open ball $B = B(x_0, r)$ contained in D*

$$P_{B^c}(x, \cdot) = \tilde{P}_{B^c}(x, \cdot) \tag{5.7}$$

for every $x \in B$. Then there exists a continuous additive functional A of X which is strictly increasing and finite on $[0, \zeta)$, such that if τ is the inverse of A, the processes $\tilde{X}$ and $\hat{X}$ (where $\hat{X}$ is defined above) are equivalent.

Proof: Note that the assumption (5.7) means that the exit distributions from open balls are equal for both processes. From the proof of Theorem 1.2 in [20] it follows that the exit distributions from all relatively compact open subsets U in D are equal, i.e., $P_{U^c}(x, \cdot) = \tilde{P}_{U^c}(x, \cdot)$ for each $x \in U$. It is easy to see then that $P_{U^c}(x, \cdot) = \tilde{P}_{U^c}(x, \cdot)$ for all x. From this it follows that for every compact subset K of D_Δ, $P_K(x, \cdot) = \tilde{P}_K(x, \cdot)$. The statement of the proposition follows from the Blumenthal-Getoor-McKean theorem (cf. [3, V(5.1)]). □

<u>Remark</u>: Whenever the conclusion of the proposition holds, we will say that $\tilde{X}$ is a time change of X.

As in Chapter 3, we say that a function $h : D \to \mathbf{R}$ is harmonic for $\tilde{X}$ if $\tilde{P}_{K^c}h(x) = h(x)$ for every compact set K in D, and every $x \in D$. Let $\tilde{\mathcal{H}}$ denote the family of all nonnegative harmonic functions for $\tilde{X}$. By $\mathcal{H}$ we denote nonnegative harmonic functions in D. These are precisely nonnegative harmonic functions for X.

PROPOSITION 5.3 *Assume that $\mathcal{H} = \tilde{\mathcal{H}}$. Then $\tilde{X}$ is a time change of X.*

PROOF: According to Proposition 5.2, it is enough to check that, for any open ball $B = B(x_0, r)$ contained in D and any $x \in B$, the measures $P_{B^c}(x, \cdot)$ and $\tilde{P}_{B^c}(x, \cdot)$ are equal. If $\bar{B}$ denotes the closure of B, then for $h \in \mathcal{H}$, $P_{B^c}h(x) = P_{\bar{B}^c}h(x)$, and similarly, since $\tilde{\mathcal{H}} = \mathcal{H}$, $\tilde{P}_{B^c}h(x) = \tilde{P}_{\bar{B}^c}h(x)$ for all x. Let us fix $x \in B$. From the previous observation it follows that $P_{B^c}h(x) = \tilde{P}_{B^c}h(x)$ for every $h \in \mathcal{H}$. Since X and $\tilde{X}$ are continuous, both measures $P_{B^c}(x, \cdot)$ and $\tilde{P}_{B^c}(x, \cdot)$ are concentrated on the boundary $\partial B = S(x_0, r)$ of B. Hence, it follows from Proposition 5.1 that these measures are equal. □

Let us note that if $\{h_n\}$ is a sequence of harmonic functions for $\tilde{X}$ which converges in $\mathcal{C}(D)$ to a function h, then h is also harmonic for $\tilde{X}$. This was proved in Proposition 3.1.

We assume now that D is a bounded Lipschitz domain in $\mathbf{R}^n$ and fix $x_0 \in D$. Let $\omega^x(A)$ denote the harmonic measure of a Borel subset A of ∂D at x. Then $x \mapsto \omega^x(A)$ is harmonic and $\omega^x(A) = P^x(X_{\zeta-} \in A)$. For a point $z \in \partial D$, let $\Delta(z, r)$ denote the "surface ball" on ∂D around z (see Section 3.1 for a precise definition) and let K denote the kernel function for D based at x_0. Then

$$K(x, z) = \lim_{n \to \infty} \frac{\omega^x(\Delta(z, 2^{-n}))}{\omega^{x_0}(\Delta(z, 2^{-n}))}, \tag{5.8}$$

and convergence is uniform on compacts. This was proved in [15, Thm. 4.1] (cf. also [4, Thm. 3.1]).

We introduce an additional hypothesis on $\tilde{X}$: We assume that $\tilde{X}_{\tilde{\zeta}-} = \lim_{t\to\tilde{\zeta}} \tilde{X}_t$ exists in ∂D.

For a Borel subset A of ∂D, let

$$h(x) = \tilde{P}^x(X_{\zeta-} \in A).$$

If K is a compact set in D,

$$\tilde{E}^x[h(X(T_{K^c}))] = \tilde{E}^x[\tilde{P}^{X(T_{K^c})}(X_{\zeta-} \in A)] =$$

$$\tilde{P}^x[X_{(T_{K^c}+\zeta\circ\theta_{K^c})-} \in A] = \tilde{P}^x[X_{\zeta-} \in A] = h(x).$$

Hence, $x \mapsto \tilde{P}^x(X_{\zeta-} \in A)$ is harmonic for $\tilde{X}$.

<u>THEOREM 5.1</u> *Let X be the Brownian motion in a bounded Lipschitz domain D, and let $\tilde{X}$ be a continuous standard Markov process in D such that $\tilde{X}_{\tilde{\zeta}-}$ exists in ∂D. Assume that*

$$P^x(X_{\zeta-} \in A) = \tilde{P}^x(\tilde{X}_{\tilde{\zeta}-} \in A) \tag{5.9}$$

for each Borel subset A of ∂D and each $x \in D$. Then $\tilde{X}$ is a time change of X.

PROOF: Let $z \in \partial D$ and $x \in D$. By (5.9) and (5.10),

$$K(x,z) = \lim_{n\to\infty} \frac{P^x(X_{\zeta-} \in \Delta(z,2^{-n}))}{P^{x_0}(X_{\zeta-} \in \Delta(z,2^{-n}))} = \lim_{n\to\infty} \frac{\tilde{P}^x(X_{\zeta-} \in \Delta(z,2^{-n}))}{\tilde{P}^{x_0}(X_{\zeta-} \in \Delta(z,2^{-n}))}.$$

Hence, $x \mapsto K(x,z)$ is harmonic for $\tilde{X}$. Let $B = B(y_0, r)$ be a ball in D and $y \in B$. Then

$$\int_{S(y_0,r)} K(x,z) P_{B^c}(y,dx) = K(y,z) = \int_{S(y_0,r)} K(x,z) \tilde{P}_{B^c}(y,dx).$$

Corollary 5.1 implies that harmonic measures $P_{B^c}(y,\cdot)$ and $\tilde{P}_{B^c}(y,\cdot)$ are equal. Proposition 5.2 completes the proof. □

We note that the theorem holds also for NTA domains. It is enough to check that the equation (5.9) is valid. But this is proved in [16, Thm. 5.5.].

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Zoran Vondraček was born in Zagreb, Yugoslavia, on July 28, 1959. He received a bachelor's degree in mathematics from the University of Zagreb in July 1982.

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